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Three-partite subamalgams of tiled orders of finite lattice type [1]

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Dedicated to the memory of Maunce Auslander

Abstract

Let D be a complete discrete valuation domain with the residue field K. We study in the paper a class of subamalgams $\Lambda^\bullet$ (1.3) of tiled D-orders Λ (1.1) by means of an integral quadratic Tits form $q_{\Lambda^\bullet} : \mathbb{Z}^m \to \mathbb{Z}$ (1.4) and a matrix problem over K defined in Section 3 by a finite stratified poset I_ρ associated with $\Lambda^\bullet$. Simple criteria for the finite lattice type of $\Lambda^\bullet$ are given in terms of the Tits form $q_{\Lambda^\bullet}$, in terms of a two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations associated to $\Lambda^\bullet$ in (4.4), and in terms of forbidden minor D-suborders of $\Lambda^\bullet$ presented in Table 1. The shape of Auslander–Reiten quiver $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ is described in Remarks 6.4. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper we assume that K is a field and D is a complete discrete valuation domain, $\mathfrak{p}$ is the unique maximal ideal of D and $K = D/\mathfrak{p}$. We denote by $F = D_0$ the field of fractions of D, and by $\mathbb{M}_m(D)$ the full $m \times m$-matrix ring with coefficients in D.

We recall that a *D-order* Λ in a finite-dimensional semisimple F-algebra C is a subring Λ of C which is a finitely generated free D-submodule of C and Λ contains an F-basis of C.

We denote by latt(Λ) the category of right Λ-lattices, that is, finitely generated right Λ-modules which are free as D-modules. It is well-known [17] that any D-order is a

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semiperfect ring and the category latt(Λ) has the finite unique decomposition property and has Auslander–Reiten sequences (see [15, 18, 19, 25, 35, 36]).

A D-order is said to be of *finite lattice type* if the category latt(Λ) has finitely many isomorphism classes of indecomposable modules. Λ is said to be of *tame lattice type* if the indecomposable Λ-lattices of any fixed D-rank form a finite set of at most one-parameter families (see [6, 27]).

It was shown by the author in [28] that the weakly positivity and the weakly non-negativity of a reduced Tits quadratic form associated to a D-order Λ constructed by a kind of glueing is a necessary and sufficient condition for the finite lattice type and for the tame lattice type of Λ, respectively. In the present paper we prove a similar result for subamalgam suborders $\Lambda^\bullet$ (1.3) of tiled D-orders (1.1).

Our main result of this paper is the characterisation given in Theorem 1.6 of a class of subamalgam D-orders of finite lattice type in terms of the associated reduced Tits quadratic form (1.4) defined below.

The subamalgam D-orders $\Lambda^\bullet$ (1.3) of tame lattice type and of polynomial growth will be characterised in the subsequent paper [31].

In order to formulate the main results we introduce some notation. We suppose that $n, n_1, n_2 > 0$ and $n_3 \geq 0$ are fixed natural numbers and Λ is a tiled D-suborder of $\mathbb{M}_n(D)$ of the form

$$(1.1)\qquad \Lambda = \left.\begin{pmatrix} D & {}_1D_2 & \dots & {}_1D_n \\ \mathfrak{p} & D & \dots & {}_2D_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & {}_{n-1}D_n \\ \mathfrak{p} & \mathfrak{p} & \dots & D \end{pmatrix}\right\} n$$

such that

(a) ${}_iD_j$ is either D or $\mathfrak{p}$,

(b) Λ admits a three-partition;

$$(1.2)\qquad \Lambda = \left(\begin{array}{c|c|c} \Lambda_1 & \mathscr{X} & \mathbb{M}_{n_1}(D) \\ \hline \mathbb{M}_{n_3\times n_1}(\mathfrak{p}) & \Lambda_3 & \mathscr{Y} \\ \hline \mathbb{M}_{n_1}(\mathfrak{p}) & \mathbb{M}_{n_1\times n_3}(\mathfrak{p}) & \Lambda_2 \end{array}\right) \begin{array}{l} \}n_1 \\ \}n_3 \\ \}n_2 \end{array}$$

where $\Lambda_2 = \Lambda_1$, $n_1 = n_2$, $n_1 + n_2 + n_3 = n$ and Λ_3 is a hereditary $n_3 \times n_3$-matrix D-order

$$\Lambda_3 = \left.\begin{pmatrix} D & D & \dots & D & D \\ \mathfrak{p} & D & \dots & D & D \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & D & D \\ \mathfrak{p} & \mathfrak{p} & \dots & \mathfrak{p} & D \end{pmatrix}\right\} n_3.$$

In particular ${}_iD_j = D$ holds in Λ for $j = n_1 + n_3 + 1, \dots, n$ and $i \leq n_1$.

Note that $1 = \varepsilon_1 + \varepsilon_3 + \varepsilon_2$, where ε_1, ε_3 and ε_2 are the matrix idempotents of Λ corresponding to the identity elements of Λ_1, Λ_3 and Λ_2, respectively.

We define the *three-partite subamalgam* of Λ to be the D-suborder

$$(1.3) \qquad \Lambda^\bullet = \{\lambda = [\lambda_{ij}];\ \varepsilon_1\lambda\varepsilon_1 - \varepsilon_2\lambda\varepsilon_2 \in \mathbb{M}_{n_1}(\mathfrak{p})\}$$

of Λ consisting of all matrices $\lambda = [\lambda_{ij}]$ of Λ such that the left upper corner $n_1 \times n_1$ submatrix $\varepsilon_1\lambda\varepsilon_1$ of λ is congruent modulo $\mathbb{M}_{n_1}(\mathfrak{p})$ to the right lower corner $n_1 \times n_1$ submatrix $\varepsilon_2\lambda\varepsilon_2$ of λ.

To any such a D-order $\Lambda^\bullet$ we associate the integral quadratic form

$$(1.4) \qquad q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \longrightarrow \mathbb{Z}$$

in the indeterminates $x_*, x_+, x_1, \ldots, x_{n_1+n_3}, \overline{x}_{n_1+1}, \ldots, \overline{x}_{n_1+n_3}$ defined by the formula

$$\begin{aligned} &q_{\Lambda^\bullet}(x_1, \ldots, x_{n_1+n_3}, \overline{x}_{n_1+1}, \ldots, \overline{x}_{n_1+n_3}, x_*, x_+) \\ &= x_*^2 + x_+^2 + \sum_{j=1}^{n_1+n_3} x_j^2 + \sum_{j=n_1+1}^{n_1+n_3} \overline{x}_j^2 \\ &+ \sum_{\substack{{}_iD_j = D \\ 1 \le i < j \le n_1+n_3}} x_i x_j + \sum_{s<t} \overline{x}_s \overline{x}_t + \sum_{\substack{{}_tD_s = D \\ n_1 < t \le n_1+n_3 < s}} x_{s-n_1-n_3}\overline{x}_t \\ &- x_+ \left(\sum_{j=1}^{n_1+n_3} x_j \right) - x_* \left(\sum_{j=1}^{n_1} x_j + \sum_{j=n_1+1}^{n_1+n_3} \overline{x}_j \right). \end{aligned}$$

Following [28] we call $q_{\Lambda^\bullet}$ the *reduced Tits quadratic form* of the order $\Lambda^\bullet$.

Given a matrix $\lambda \in \mathbb{M}_n(D)$ we define the *reflection transpose* of λ to be the transpose matrix

$$rt(\lambda) \in \mathbb{M}_n(D)$$

of λ with respect to the non main diagonal.

Given any D-order Λ we define the *reflection transpose* of Λ (resp. of $\Lambda^\bullet$) to be the D-order

$$(1.5) \qquad rt(\Lambda) = \{rt(\lambda);\ \lambda \in \Lambda\} \quad (\text{resp.} \quad rt(\Lambda^\bullet) = \{rt(\lambda);\ \lambda \in \Lambda^\bullet\}\,)$$

It is easy to see that

$$rt(\Lambda^\bullet) = rt(\Lambda)^\bullet$$

and the map $\lambda \mapsto rt(\lambda)$ defines the ring anti-isomorphisms $\Lambda \xrightarrow{\simeq} rt(\Lambda)$ and $\Lambda^\bullet \xrightarrow{\simeq}$, $rt(\Lambda^\bullet)$.

One of our main results of this paper is the following theorem.

Theorem 1.6. *Assume that D is a complete discrete valuation domain. Let Λ be a three-partite D-order* (1.2) *as above and let $\Lambda^\bullet$ be the subamalgam* (1.3) *of Λ, where $\Lambda_1 = \Lambda_2$, Λ_3 and n, $n_1 = n_2$, n_3 are positive integers as above. The following conditions are equivalent*:

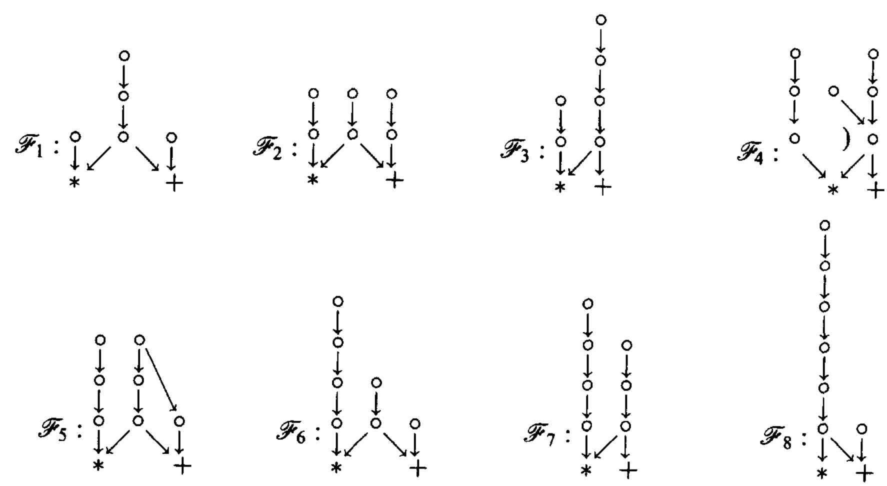
(a) *The D-order $\Lambda^\bullet$ is of finite lattice type.*

(b) *The integral reduced Tits quadratic form $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z}$* (1.4) *is weakly positive, that is, $q_{\Lambda^\bullet}(z) > 0$ for any non-zero vector $z \in \mathbb{N}^{n_1+2n_3+2}$.*

(c) *The D-order Λ_1 is hereditary of the form*

$$(1.7)\qquad \begin{pmatrix} D & D & \dots & D & D \\ \mathfrak{p} & D & \dots & D & D \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & D & D \\ \mathfrak{p} & \mathfrak{p} & \dots & \mathfrak{p} & D \end{pmatrix}$$

and the two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations associated to $\Lambda^\bullet$ in (4.4) *does not contain as a two-peak subposet with zero-relations any of the following eight forms*:



where the dotted line in $\mathscr{F}_4$ means a zero-relation.

(d) *The D-order Λ_1 is hereditary of the form* (1.7) *and the three-partite orders $\Lambda^\bullet$ and $rt(\Lambda)^\bullet$ do not contain three-partite minor D-suborders being dominated by any of the 14 three-partite D-orders listed in Table* 1.

If $1 \leq i_1 < \cdots < i_s \leq n_1$, we say that the order Δ is an *$(i_1, \dots, i_s)$-minor D-suborder* of Λ_1 if Δ is obtained from Λ_1 by omitting the i_jth row and the i_jth column for $j = 1, \dots, s$.

A three-partite order Γ is said to be a *three-partite minor D-suborder* of $\Lambda^\bullet$ if Γ is a minor D-suborder of $\Lambda^\bullet$ obtained by omitting rows and columns simultaneously in parts Λ_1 and Λ_2, that is, we omit any ith row and any ith column of $\Lambda^\bullet$, where $1 \le i \le n_1$, and simultaneously we omit the (n_3+i)th row and the (n_3+i)th column of $\Lambda^\bullet$.

A three-partite order $\Lambda^\bullet$ (1.3) is said to be *dominated* by a three-partite order $\overline{\Lambda}^\bullet$ if $\Lambda^\bullet$ is a three-partite D-suborder $\overline{\Lambda}^\bullet$ of the same size (1.2) and $\Lambda_1 = \overline{\Lambda}_1$, $\Lambda_2 = \overline{\Lambda}_2$, $\Lambda_3 = \overline{\Lambda}_3$, $\mathscr{X} \subseteq \overline{\mathscr{X}}$, $\mathscr{Y} \subseteq \overline{\mathscr{Y}}$ (see [36, p.69]).

In Section 3 and in (5.1) we associate with $\Lambda^\bullet$ a family

$$* : G_{\Lambda^\bullet}(v) \times \mathscr{M}_{\Lambda^\bullet}(v) \to \mathscr{M}_{\Lambda^\bullet}(v), \quad v \in \mathbb{N}^{n_1+2n_3+2},$$

of parabolic algebraic groups $G_{\Lambda^\bullet}(v) = \mathbf{G}_v^{(I^{*+}_{\Lambda^\bullet}, 3_{\Lambda^\bullet})}$ acting on irreducible algebraic K-varieties $\mathscr{M}_{\Lambda^\bullet}(v) = \mathbf{Mat}_v^{(I^{*+}_{\Lambda^\bullet}, 3_{\Lambda^\bullet})}$. We show in Corollary 5.2 that $\Lambda^\bullet$ is of finite lattice type if and only if the number of $G_{\Lambda^\bullet}(v)$-orbits in $\mathscr{M}_{\Lambda^\bullet}(v)$ is finite, or equivalently, if and only if $\dim G_{\Lambda^\bullet}(v) \ge \dim \mathscr{M}_{\Lambda^\bullet}(v)$ for all $v \in \mathbb{N}^{n_1+2n_3+2}$, where dim is the variety dimension.

In case latt(Λ) is of finite representation type a construction of Auslander–Reiten quiver of the category latt($\Lambda^\bullet$) is presented in Theorem 6.1. In particular, it is shown there that the map $X \mapsto \mathbf{cdn}\,\mathbb{H}(X)$ defines a bijection

$$\text{(1.8)} \qquad \operatorname{ind}\operatorname{latt}(\Lambda^\bullet) \longrightarrow \mathscr{R}^+_{\Lambda^\bullet} \setminus \{\varepsilon_1, \dots, \varepsilon_{n_1+n_3}, \overline{\varepsilon}_{n_1+1}, \dots, \overline{\varepsilon}_{n_1+n_3}, \varepsilon_*, \widehat{\varepsilon}, \widehat{\varepsilon}_{n_1+1}, \dots, \widehat{\varepsilon}_{n_1+n_3}\}$$

between the set ind latt($\Lambda^\bullet$) of isomorphism classes of indecomposable lattices in latt($\Lambda^\bullet$) and the set

$$\text{(1.9)} \qquad \mathscr{R}^+_{\Lambda^\bullet} = \{v \in \mathbb{N}^{n_1+2n_3+2};\ q_{\Lambda^\bullet}(v) = 1\}$$

of *positive roots* of the quadratic form $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z}$ (1.4) being distinct from the roots $\varepsilon_1, \dots, \varepsilon_{n_1+n_3}$, $\overline{\varepsilon}_{n_1+1}, \dots, \overline{\varepsilon}_{n_1+n_3}$, ε_*, $\widehat{\varepsilon}$, $\widehat{\varepsilon}_{n_1+1}, \dots, \widehat{\varepsilon}_{n_1+n_3}$, where $\mathbb{H}$ is the reduction functor (4.1), $\mathbf{cdn}\,Y \in \mathbb{N}^{n_1+2n_3+2}$ is the coordinate vector defined by the formula (3.1), $\varepsilon_1, \dots, \varepsilon_{n_1+n_3}, \overline{\varepsilon}_{n_1+1}, \dots, \overline{\varepsilon}_{n_1+n_3}, \varepsilon_*, \varepsilon_+$ is the standard basis of the free abelian group $\mathbb{Z}^{n_1+2n_3+2}$, $\widehat{\varepsilon}_{n_1+s} = \varepsilon_{n_1+s} + \widehat{\varepsilon}$, $\widehat{\varepsilon} = \sum_{c \in m(\Lambda_1)} \varepsilon_c$ and $m(\Lambda_1)$ is the subset of $\{1, \dots, n_1\}$ consisting of all elements $t \le n_1$ such that there is no $m < t$ with ${}_mD_t = D$.

Main results of this paper were presented on an AMS-IMS-SIAM Joint Summer Conference "Trends in the Representation Theory of Finite Dimensional Algebras" at the University of Washington, Seattle, in July 1997.

2. Right multipeak posets with zero-relations and their socle projective representations

We recall from [37; 25, Ch. 13; 29] that the study of tiled D-orders is reduced to the study of representations of infinite posets having a unique maximal element by means of a covering type technique. A similar idea applies in the study of some categories of Cohen–Macaulay modules, lattices over pull-back orders and categories of abelian groups (see [1–3, 9, 36]).

Table 1

Minimal three-partite subamalgams of tiled D-orders of infinite lattice type up to domination

We use here the notation introduced in Section 1, that is, we present three-partite tiled D-orders Γ_j, and the three-partition is indicated by vertical and horizontal lines. The subamalgam D-order $\Gamma_j^\bullet$ is obtained from Γ_j according to the formula (1.3) by identifying modulo $\mathfrak{p} = \mathrm{rad}(D)$ the left upper corner block of Γ_j with the right lower corner block of Γ_j.

Type $\mathscr{F}_1$:

$$\Gamma_1 = \left(\begin{array}{ccc|c|ccc} D & D & D & \mathfrak{p} & D & D & D \\ \mathfrak{p} & D & D & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 1, \qquad \Gamma_2 = \left(\begin{array}{ccc|cc|ccc} D & D & D & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 2$$

Type $\mathscr{F}_2$:

$$\Gamma_3 = \left(\begin{array}{cc|cc|cc} D & D & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 2, \qquad \Gamma_4 = \left(\begin{array}{cc|ccc|cc} D & D & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 3,$$

$$\Gamma_5 = \left(\begin{array}{cc|cccc|cc} D & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 4$$

Type $\mathscr{F}_3$:

$$\Gamma_6 = \left(\begin{array}{cccc|cc|cccc} D & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 2$$

Type $\mathscr{F}_4$:

$$\Gamma_7 = \left(\begin{array}{ccc|cccc|ccc} D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 4, \qquad \Gamma_8 = \left(\begin{array}{ccc|ccc|ccc} D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) \quad n_3 = 3,$$

Table 1. Contd.

Type $\mathscr{F}_5$:

$$\Gamma_9 = \left(\begin{array}{ccc|cccc|ccc}
D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4, \quad
\Gamma_{10} = \left(\begin{array}{ccc|ccc|ccc}
D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 3$$

Type $\mathscr{F}_6$:

$$\Gamma_{11} = \left(\begin{array}{cc|ccccc|cc}
D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 5, \quad
\Gamma_{12} = \left(\begin{array}{cc|ccc|ccc}
D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4$$

Type $\mathscr{F}_7$:

$$\Gamma_{13} = \left(\begin{array}{ccc|cccc|ccc}
D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4$$

Type $\mathscr{F}_8$:

$$\Gamma_{14} = \left(\begin{array}{cccccc|c|cccccc}
D & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 1$$

Our principal idea in proving the main theorems of the paper is a reduction of the problem for lattices over three-partite subamalgams of tiled D-orders to a corresponding problem for K-linear socle projective representations of two-peak posets (that is, having exactly two maximal elements) with zero-relations and studied in [24], where $K = D/\mathfrak{p}$. Our reduction extends the reduction given in [28, Section 2] and will involve the reduction functors defined in [8, 19], and the covering technique for bipartite stratified posets developed by the author in [22, 24].

The reader is referred to the author's survey articles [22, 29] for an elementary explanation of the role of peak posets I, peak I-spaces and their relevance to amalgamation of D-orders and their representation type.

Throughout we shall denote by $(I; \preceq)$ a finite *poset*, that is, a finite partially ordered set $(I; \preceq)$ with the partial order $\preceq$. We shall write $i \prec j$ if $i \preceq j$ and $i \neq j$. For the sake of simplicity we write I instead of $(I, \preceq)$. We denote by $\max I$ the set of all maximal elements of I and I will be called an *r-peak poset* if $|\max I| = r$.

Given a poset I we denote by KI the incidence algebra of I [25], that is, the subalgebra of the full matrix algebra $\mathbb{M}_I(K)$ consisting of all $I \times I$ square matrices $\lambda = [\lambda_{pq}]_{p,q \in I}$ such that $\lambda_{pq} = 0$ if $p \not\preceq q$ in $(I; \preceq)$.

For $i \preceq j$ we denote by $e_{ij} \in KI$ the matrix having 1 at $(i\text{-}j)$th position and zero elsewhere. Given j in I we denote by $e_j = e_{jj}$ the standard primitive idempotent of KI corresponding to j.

The algebra KI is basic and the standard matrix idempotents e_i, $i \in I$, form a complete set of primitive orthogonal idempotents of KI. Moreover, KI is of finite global dimension and the right socle of KI is isomorphic to a direct sum of copies of the right ideals $e_p KI$, $p \in \max I$, called the *right peak ideals* of KI (see [26]).

We shall denote by $\mathrm{mod}_{sp}(KI)$ the category of *socle projective right KI-modules* [21], that is, the full subcategory of $\mathrm{mod}(KI)$ consisting of modules X such that the socle $\mathrm{soc}(X)$ of X is projective and isomorphic to a direct sum of copies of the right ideals $e_p KI$, $p \in \max I$.

In our definition of a main reduction functor introduced in Section 4 we shall also need a concept of a poset with zero-relations and its incidence K-algebra.

Definition 2.1. A *zero-relation* in a poset I is a pair (i_0, j_0) of elements of I such that $i_0 \prec j_0$.

A set of *zero-relations* in I is a set $\mathfrak{Z}$ satisfying the following two conditions:

(Z1) $\mathfrak{Z}$ consists of zero-relations (i_0, j_0) of I,

(Z2) If $(i_0, j_0) \in \mathfrak{Z}$ and $i_1 \preceq i_0 \preceq j_0 \preceq j_1$ then $(i_1, j_1) \in \mathfrak{Z}$.

A *right multipeak (or precisely r-peak) poset with zero-relations* is a pair $(I, \mathfrak{Z})$, where I is a poset, $r = |\max I|$, and $\mathfrak{Z}$ is a set of zero-relations satisfying the following condition (see [23, p. 118]):

(Z3) For every $i \in I \setminus \max I$ there exists $p \in \max I$ such that $(i, p) \notin \mathfrak{Z}$. In case the set $\mathfrak{Z}$ is empty we shall write I instead of $(I, \mathfrak{Z})$.

A right multipeak poset $(I', \mathfrak{Z}')$ with zero-relations is said to be a peak subposet of $(I, \mathfrak{Z})$ if I' is a subposet of I, $\mathfrak{Z}'$ is the restriction of $\mathfrak{Z}$ to I' and $\max I' = I' \cap (\max I)$.

Definition 2.2. Let K be a field and let $(I, \mathfrak{Z})$ be a right r-peak poset with zero-relations. The *incidence K-algebra* of $(I, \mathfrak{Z})$ is the K-algebra

$$(2.1) \qquad K(I, \mathfrak{Z}) = \{\lambda = [\lambda_{ij}]_{i,j \in I} \in KI;\ \lambda_{ij} = 0, \text{ for } (i,j) \in \mathfrak{Z}\} \subseteq KI$$

consisting of all $I \times I$ square matrices $\lambda = [\lambda_{ij}]_{i,j \in I} \in \mathbb{M}_I(K)$ such that $\lambda_{ij} = 0$ if $i \not\preceq j$ in $(I; \preceq)$, or if $(i,j) \in \mathfrak{Z}$. The addition in $K(I, \mathfrak{Z})$ is the usual matrix addition, whereas the multiplication of two matrices $\lambda = [\lambda_{ij}]_{i,j\in I}$ and $\lambda' = [\lambda'_{ij}]_{i,j\in I}$ in $K(I,\mathfrak{Z})$ is the matrix $\lambda'' = [\lambda''_{ij}]_{i,j\in I}$, where

$$\lambda''_{ij} = \begin{cases} \sum\limits_{i\preceq s\preceq j} \lambda_{is}\lambda'_{sj} & \text{if } i \preceq j \text{ and } (i,j) \notin \mathfrak{Z}, \\ 0 & \text{if } i \not\preceq j \text{ or } (i,j) \in \mathfrak{Z}. \end{cases}$$

Note that in case the set $\mathfrak{Z}$ is not empty the algebra $K(I,\mathfrak{Z}) \subseteq KI$ is not a subalgebra of the incidence K-algebra $KI \subseteq \mathbb{M}_I(K)$ of the poset I. In case the set $\mathfrak{Z}$ is empty we get $KI = K(I,\mathfrak{Z})$.

It is easy to see that the incidence algebra $K(I,\mathfrak{Z})$ is basic and the standard matrix idempotents e_i, $i \in I$, form a complete set of primitive orthogonal idempotents of $K(I,\mathfrak{Z})$.

We shall denote by $\operatorname{mod}_{\mathrm{sp}} K(I,\mathfrak{Z})$ the category of *socle projective right $K(I,\mathfrak{Z})$-modules*, that is, the full subcategory of $\operatorname{mod} K(I,\mathfrak{Z})$ consisting of modules X such that the socle $\operatorname{soc}(X)$ of X is projective and isomorphic to a direct sum of copies of the right ideals $e_pK(I,\mathfrak{Z})$, $p \in \max I$ (see [21]).

Lemma 2.4. *Let K be a field and $K(I,\mathfrak{Z})$ be the incidence K-algebra of a right peak poset $(I,\mathfrak{Z})$ with zero-relations.*

(a) *The algebra $K(I,\mathfrak{Z})$ is a factor K-algebra of KI modulo the ideal generated by all matrices $e_{ij} \in KI$ such that $(i,j) \in \mathfrak{Z}$.*

(b) *The right socle of the algebra $K(I,\mathfrak{Z})$ is isomorphic to a direct sum of copies of the right ideals $e_pK(I,\mathfrak{Z})$, $p \in \max I$, called the right peak ideals of $K(I,\mathfrak{Z})$.*

(c) *The global dimension of $K(I,\mathfrak{Z})$ is finite.*

(d) *The category $\operatorname{mod}_{\mathrm{sp}} K(I,\mathfrak{Z})$ is closed under extensions, direct sums and summands. It has Auslander–Reiten sequences, source maps and sink maps, enough relative projective and enough relative injective objects.*

Proof. The statement (a) is an easy observation. The statement (c) follows from (a) by standard arguments like in the proof of [26, Lemma 2.1]. It follows from the condition (Z3) in Definition 2.1 that the algebra $K(I,\mathfrak{Z})$ is a right multipeak ring in the sense of [21, Section 1]. Then (b) follows like in [20, Proposition 2.2] or in the proof of [25, Corollary 17.44]. The statement (d) is a consequence of [16]. □

It is well-known that any finite-dimensional module X over $K(I,\mathfrak{Z})$ can be identified with the K-linear representation of $(I,\mathfrak{Z})$, that is, the system

(2.5) $\quad X = (X_i, {}_jh_i)_{i,j\in I, i\prec j}$

where $X_i = Xe_i$, ${}_jh_i : X_i \to X_j$ is the K-linear map defined by the multiplication by $e_{ij} \in KI$ (see [32, Section 2] for more details).

The module X is socle projective if and only if X viewed as a K-linear representation $X=(X_i, {}_jh_i)_{i,j\in I, i\prec j}$ of $(I,\mathfrak{Z})$ is socle projective, that is, if

$$\bigcap_{p\in \max I} \operatorname{Ker} {}_ph_i = 0$$

for any $i\in I\setminus \max I$ (see [21]). It is often useful to deal with filtered forms of socle projective K-linear representations of $(I,\mathfrak{Z})$. For this purpose we introduce the following definition extending that one given in [26] for posets without zero-relations.

Definition 2.6. Let K be a field and let $(I,\mathfrak{Z})$ be a right multipeak poset with zero-relations. A *peak* $(I,\mathfrak{Z})$*-space* (or *a filtered socle projective representation* of $(I,\mathfrak{Z})$) over the field K is the system

$$\mathbf{M}=(M_j)_{j\in I}$$

of finite-dimensional K-vector spaces M_j satisfying the following four conditions:

(a) For any $j\in I$ the K-space M_j is a K-subspace of the space

$$M^{\bullet}=\bigoplus_{p\in \max I} M_p.$$

(b) The inclusion $M_p\subseteq M^{\bullet}$ is the usual p-coordinate embedding for any $p\in \max I$.

(c) $\pi_j(M_i)\subseteq M_j$ for all $i\prec j$ in I, where $\pi_j: M^{\bullet}\to M^{\bullet}$ is the composed K-linear endomorphism

$$M^{\bullet}\xrightarrow{\pi_j'}\bigoplus_{\substack{j\preceq p\in \max I\\ (i,p)\notin\mathfrak{Z}}} M_p \hookrightarrow M^{\bullet}$$

of $M^{\bullet}$ and π_j' is the direct summand projection.

(d) If $p\in \max I$ and either $i\not\preceq p$ or $i\prec p$ and $(i,p)\in\mathfrak{Z}$ then $\pi_p(M_i)=0$.

A morphism $f:\mathbf{M}\to\mathbf{M}'$ from $\mathbf{M}$ to $\mathbf{M}'$ is a system $f=(f_p)_{p\in \max I}$ of K-linear maps $f_p: M_p\to M_p'$, $p\in \max I$, such that $(\bigoplus_{p\in \max I} f_p)(M_j)\subseteq M_j'$ for all $j\in I$.

We denote by $(I,\mathfrak{Z})$-spr the *category of peak I-spaces* (or filtered socle projective representations of $(I,\mathfrak{Z})$) over the field K. The direct sum and the indecomposablity in the category $(I,\mathfrak{Z})$-spr are defined in an obvious way.

In case the set $\mathfrak{Z}$ is empty the category $(I,\mathfrak{Z})$-spr is the category I-spr of peak I-spaces (or socle projective K-linear representations of I introduced in [26].

It is easy to see that $(I,\mathfrak{Z})$-spr is an additive category with the finite unique decomposition property [25, Section 1.1], and the K-linear functor

(2.7) $\quad \rho:(I,\mathfrak{Z})\text{-spr}\xrightarrow{\simeq}\operatorname{mod}_{sp} K(I,\mathfrak{Z})$

$\mathbf{M} \mapsto \widehat{\mathbf{M}} = (M_j;\ {}_j\pi_i)_{i \prec j}$, is an equivalence of categories, where ${}_j\pi_i : M_i \to M_j$ is a unique K-linear map making the diagram

$$\begin{array}{ccc} M_i & \subseteq & M^\bullet \\ \big\downarrow {\scriptstyle {}_j\pi_i} & & \big\downarrow {\scriptstyle \pi_j} \\ M_j & \subseteq & M^\bullet \end{array}$$

commutative. The quasi-inverse of ρ is the restriction to the category $\mathrm{mod}_{\mathrm{sp}}\, K(I, \mathfrak{Z})$ of the *adjustment functor* (see [21; 16; 25, (11.32); 26]).

(2.8) $\quad \Theta : \mathrm{mod}\, K(I, \mathfrak{Z}) \longrightarrow K(I, \mathfrak{Z})\text{-spr}$

associating to $X = (X_i, {}_jh_i)_{i,j \in I, i \prec j}$ the peak $(I, \mathfrak{Z})$-space $\mathbf{M}(X) = (M(X)_j)_{j \in I}$, where

$$M(X)_j = \begin{cases} X_j & \text{for } j \in \max\ I, \\ \mathrm{Im}\,[({}_ph_j)_{p \in \max I} : X_j \to \bigoplus\limits_{p \in \max\ I} X_p] & \text{for } j \in I \setminus \max\ I. \end{cases}$$

The discussion above together with Lemma 2.4 (d) yields.

Corollary 2.9. (a) *The functor* $\rho : (I, \mathfrak{Z})\text{-spr} \longrightarrow \mathrm{mod}_{\mathrm{sp}} K(I, \mathfrak{Z})$ (2.7) *is an equivalence of categories and the functor* (2.8) *restricted to* $\mathrm{mod}_{\mathrm{sp}} K(I, \mathfrak{Z})$ *is the quasi-inverse of* ρ.

(b) *The category* $(I, \mathfrak{Z})$-spr *is an additive K-category. Moreover* $(I, \mathfrak{Z})$-spr *has Auslander-Reiten sequences, source maps, sink maps, enough relative projective objects and enough relative injective objects* [16].

Following [16; 26, (3.1)] we associate to any r-peak poset $(I, \mathfrak{Z})$ with zero-relations the integral *Tits quadratic form* $q_{(I,\mathfrak{Z})} : \mathbb{Z}^I \to \mathbb{Z}$ defined by the formula

$$(2.10) \qquad q_{(I,\mathfrak{Z})}(z) = \sum_{j \in I} z_j^2 + \sum_{\substack{i \prec j \notin \max I \\ (i,j) \in \mathfrak{Z}}} z_i z_j - \sum_{p \in \max\ I} \sum_{\substack{i \prec p \\ (i,p) \in \mathfrak{Z}}} z_i z_p.$$

3. Prinjective modules and a family of algebraic groups acting on algebraic varieties

Throughout this section we suppose that

$$I = \{1, \ldots, n,\ p_1, \ldots, p_r\}, \quad \max\ I = \{p_1, \ldots, p_r\}.$$

Moreover, we suppose that the order relation $\prec$ in I is such that $i \prec j$ implies that $i < j$ in the natural order. We can always achieve this by a suitable renumbering of the elements in I. Given $j \in I$ we set

$$j^\nabla = \{i \in I \mid i \preceq j\} \quad \text{and} \quad I^- = I \setminus \max\ I.$$

The incidence algebra $K(I,\mathfrak{Z})$ has an obvious bipartition

$$K(I,\mathfrak{Z}) = \begin{pmatrix} K(I^-,\mathfrak{Z}^-) & M \\ 0 & B \end{pmatrix},$$

where $B = \bigoplus_{p\in\max I} e_pK = K\times K\times\cdots\times K$ ($|\max I|$-times), the K-space $M = \bigoplus_{p\in\max I}\bigoplus_{\substack{i\prec p\\ (i,p)\notin\mathfrak{Z}}} e_{ip}K$ is viewed as a $K(I^-,\mathfrak{Z}^-)$-B-bimodule in the obvious way and multiplication is given by the matrix multiplication. Therefore the category $\operatorname{prin} K(I,\mathfrak{Z})$ of *prinjective* right $K(I,\mathfrak{Z})$-modules is defined (see [16]), and the study of socle projective $K(I,\mathfrak{Z})$-modules reduces to the study of the category $\operatorname{prin} K(I,\mathfrak{Z})$ via the functor (3.2) defined below.

It is easy to prove like in the proof of [25, Proposition 11.32] that a module X in $\operatorname{mod} K(I,\mathfrak{Z})$ is prinjective if and only if there exists a short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

in $\operatorname{mod} K(I,\mathfrak{Z})$), where P_0, P_1 are projective $K(I,\mathfrak{Z})$-modules and P_1 is semisimple of the form $P_1 = \bigoplus_{p\in\max I}(e_pK(I,\mathfrak{Z}))^{t_p}$, $t_p \geq 0$.

Let us present a useful interpretation of modules in $\operatorname{prin} K(I,\mathfrak{Z})$ in terms of partitioned matrices with coefficients in the field K similar to that one in [26]. For this purpose given a module X in $\operatorname{mod} K(I,\mathfrak{Z})$ we define the *coordinate vector* $\mathbf{cdn}(X) \in \mathbb{N}^I$ by the following formula (see [21, 16, 26]):

$$(3.1)\qquad (\mathbf{cdn}(X))(j) = \begin{cases} \dim_K(X_j) & \text{for } j\in\max\ I,\\ \dim_K(\operatorname{top}X)e_j & \text{for } j\in I\setminus\max\ I.\end{cases}$$

We view $\mathbf{cdn}(X)$ as a map $\mathbf{cdn}(X): I\to\mathbb{N}$. Note that the projective cover $P(X)$ of X has the form

$$P(X) = \bigoplus_{j\in I^-}(e_jK(I,\mathfrak{Z}))^{(\mathbf{cdn}X)(j)}$$

if X is an indecomposable object in $\operatorname{prin} K(I,\mathfrak{Z})$ or in $(I,\mathfrak{Z})$-spr $\cong \operatorname{mod}_{sp}(K(I,\mathfrak{Z}))$, and X is not isomorphic to the simple projective modules $e_{p_1}K(I,\mathfrak{Z}),\ldots,e_{p_r}K(I,\mathfrak{Z})$.

We define the adjustment functor

$$(3.2)\qquad \Theta : \operatorname{prin}\ K(I,\mathfrak{Z}) \to \operatorname{mod}_{sp}\ K(I,\mathfrak{Z})$$

to be the composition of the equivalence (2.7) with the adjustment functor (2.8) restricted to the category $\operatorname{prin} K(I,\mathfrak{Z})$.

The discussion above together with results in [16] yields (see also [25, Section 11. 6; 26]).

Corollary 3.3. (a) *The adjustment functor* Θ (3.2) *is full dense and* $\operatorname{Ker}\Theta = [e_jK(I^-,\mathfrak{Z}^-);\ j\in I^-]$, *that is,* $\operatorname{Ker}\Theta$ *consists of all maps in* $\operatorname{prin} K(I,\mathfrak{Z})$ *having a factorisation through a direct sum of copies of the projective* $K(I^-,\mathfrak{Z}^-)$*-modules* $e_jK(I^-,\mathfrak{Z}^-)$, $j\in I^- = I\setminus\max\ I$.

(b) *If X is an indecomposable module in* $\operatorname{prin} K(I,\mathfrak{J})$ *and* $\boldsymbol{\Theta}(X) \neq 0$ *then* $\mathbf{cdn}\, X = \mathbf{cdn}\, \boldsymbol{\Theta}(X)$.

(c) *The adjustment functor* $\boldsymbol{\Theta}$ (3.2) *preserves and reflects finite representation type, and induces an equivalence of categories* $\operatorname{prin} K(I,\mathfrak{J})/[e_j K(I^-,\mathfrak{J}^-);\ j \in I^-] \cong (I,\mathfrak{J})\text{-spr}$.

Fix a vector $v \in \mathbb{N}^I$. Following [26], we associate with v and $(I,\mathfrak{J})$ the affine K-space

$$(3.4) \qquad \mathbf{prin}_v^{K(I,\mathfrak{J})} = \operatorname{Hom}_{K(I,\mathfrak{J})}(P(v), Q(v)),$$

where

$$P(v) = \bigoplus_{j \in I^-} (e_j K(I,\mathfrak{J}))^{v(j)}, \quad Q(v) = \bigoplus_{p \in \max I} E_{K(I,\mathfrak{J})}(e_p K(I,\mathfrak{J}))^{v(p)}.$$

It is clear that $\mathbf{prin}_v^{K(I,\mathfrak{J})}$ is an irreducible algebraic K-variety (in the Zariski topology). Consider the natural algebraic group action

$$(3.5) \qquad \star : \mathbf{G}(v) \times \mathbf{prin}_v^{K(I,\mathfrak{J})} \longrightarrow \mathbf{prin}_v^{K(I,\mathfrak{J})},$$

where $\mathbf{G}(v) = \operatorname{Aut}_{K(I,\mathfrak{J})}(P(v)) \times \operatorname{Aut}_{K(I,\mathfrak{J})}(Q(v))$. Note that there is a bijection between the isomorphism classes of modules in $\operatorname{prin} K(I,\mathfrak{J})$ with $\mathbf{cdn}(X) = v$ and the $\mathbf{G}(v)$-orbits in $\mathbf{prin}_v^{K(I,\mathfrak{J})}$ given by attaching to the prinjective module X the composed $K(I,\mathfrak{J})$-homomorphism (see also [32, (3.9)])

$$(3.6) \qquad f_X = [P(v) \overset{u}{\hookrightarrow} P(X) \overset{w}{\to} \boldsymbol{\Theta}(X) \overset{u'}{\hookrightarrow} E_{K(I,\mathfrak{J})}(\boldsymbol{\Theta}(X)) \cong Q(v)],$$

where u and u' are the natural embeddings and w is the natural epimorphism.

We recall that given $p \in \max I$, the injective envelope $E_{K(I,\mathfrak{J})}(e_p K(I,\mathfrak{J}))$ of $e_p K(I,\mathfrak{J})$ (viewed as a K-linear representation of the poset $(I,\mathfrak{J})$) with zero-relations is the constant diagram having the space K over all $j \preceq p$, $(i,p) \notin \mathfrak{J}$ and the space zero elsewhere. It follows that

$$\operatorname{Hom}_{K(I,\mathfrak{J})}(e_i K(I,\mathfrak{J}), E_{K(I,\mathfrak{J})}(e_p K(I,\mathfrak{J}))) \cong \begin{cases} K & \text{if } i \preceq p \text{ and } (i,p) \notin \mathfrak{J}, \\ 0 & \text{if } i \not\preceq p, \text{ or } (i,p) \in \mathfrak{J} \end{cases}$$

where $p \in \max I$. Therefore the variety $\mathbf{prin}_v^{K(I,\mathfrak{J})}$ can be identified with the variety $\mathbf{Mat}_v^{(I,\mathfrak{J})}$ of all partitioned matrices of the form (compare with [26])

$$(3.7) \qquad A = \begin{array}{c|c|c|c|l} \hline A_{1p_1} & A_{2p_1} & \cdots & A_{np_1} & \}v(p_1) \\ \hline \vdots & \vdots & \cdots & \vdots & \quad\vdots \\ \hline A_{1p_r} & A_{2p_r} & \cdots & A_{np_r} & \}v(p_r) \\ \hline \underbrace{\phantom{A_{1p}}}_{v(1)} & \underbrace{\phantom{A_{2p}}}_{v(2)} & \cdots & \underbrace{\phantom{A_{np}}}_{v(n)} & \end{array}$$

with coefficients in K, where $A_{ip_t} = 0$ if either $i \not\preceq p_t$, or $(i,p_t) \in \mathfrak{J}$, $t = 1,\ldots,r$.

Under the identification above the group $\mathbf{G}(v)$ is isomorphic to the group

$$(3.8)\qquad \mathbf{G}_v^{(I,\mathfrak{Z})} = H_v^{(I,\mathfrak{Z})} \times \mathrm{Gl}(v(p_1),K) \times \cdots \times \mathrm{Gl}(v(p_r),K),$$

where $H_v^{(I,\mathfrak{Z})}$ is a group consisting of all matrices of the form

$$(3.9)\qquad h = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ 0 & h_{22} & \dots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_{nn} \end{bmatrix} \in \mathrm{Gl}(v(1)+\cdots+v(n),\, K),$$

where $h_{ii} \in \mathrm{Gl}(v(i),K)$ and $h_{ij} = 0$ if either i and j are not comparable in I^-, or $(i,j) \in \mathfrak{Z}$. We suppose that the order in I is such that $i \prec j$ implies that $i < j$ in the natural order.

The multiplication of two matrices $h = [h_{ij}]$ and $h' = [h'_{ij}]$ in $H_v^{(I,\mathfrak{Z})}$ is the matrix $h'' = [h''_{ij}]$, where

$$(3.10)\qquad (h \cdot h')_{ij} = h''_{ij} = \begin{cases} \sum\limits_{i \preceq s \preceq j} h_{is}h'_{sj} & \text{if } i \preceq j \text{ and } (i,j) \notin \mathfrak{Z}, \\ 0 & \text{if } i \npreceq j \text{ or } (i,j) \in \mathfrak{Z}. \end{cases}$$

Note that in case the set $\mathfrak{Z}$ is not empty the group $H_v^{(I,\mathfrak{Z})} \subseteq \mathrm{Gl}(v(1)+\cdots+v(n),\, K)$ is not a subgroup of the group $\mathrm{Gl}(v(1)+\cdots+v(n),\, K)$.

We define an algebraic group action

$$(3.11)\qquad \bullet : \mathbf{G}_v^{(I,\mathfrak{Z})} \times \mathbf{Mat}_v^{(I,\mathfrak{Z})} \longrightarrow \mathbf{Mat}_v^{(I,\mathfrak{Z})}$$

by the formula $(h, g_1, \dots, g_p) \bullet A = \mathrm{diag}(g_1, \dots, g_p) A \circ h^{-1}$, where $h \in H_v^{(I,\mathfrak{Z})}$, $g_j \in \mathrm{Gl}(v(p_j),K)$, $A \circ h^{-1} = A'$ is a partitioned matrix of the form (3.7) with $A'_{sp} = (Ah^{-1})_{sp}$, if $s \preceq p$ and $(s,p) \notin \mathfrak{Z}$, and $A'_{sp} = 0$, if $s \npreceq p$ or $(s,p) \in \mathfrak{Z}$.

Proposition 3.12. *Let K be a field, let $(I,\mathfrak{Z})$ be a right multipeak poset with zero-relations and* $\max I = \{p_1, \dots, p_r\}$.

(a) *Under the notation introduced above,* $\mathbf{prin}_v^{K(I,\mathfrak{Z})}$ *and* $\mathbf{Mat}_v^{(I,\mathfrak{Z})}$ *are irreducible algebraic K-varieties in Zariski topology,* $\mathbf{G}(v)$ *and* $\mathbf{G}_v^{(I,\mathfrak{Z})}$ *are parabolic algebraic groups, there exist a K-variety isomorphism* $\mathbf{prin}_v^{K(I,\mathfrak{Z})} \xrightarrow{\sim} \mathbf{Mat}_v^{(I,\mathfrak{Z})}$ *and an algebraic group isomorphism* $\mathbf{G}(v) \xrightarrow{\sim} \mathbf{G}_v^{(I,\mathfrak{Z})}$ *such that the following diagram is commutative*

$$\begin{array}{ccc} \mathbf{G}(v) \times \mathbf{prin}_v^{K(I,\mathfrak{Z})} & \xrightarrow{*} & \mathbf{prin}_v^{K(I,\mathfrak{Z})} \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathbf{G}_v^{(I,\mathfrak{Z})} \times \mathbf{Mat}_v^{(I,\mathfrak{Z})} & \xrightarrow{\bullet} & \mathbf{Mat}_v^{(I,\mathfrak{Z})} \end{array}$$

for any coordinate vector $v \in \mathbb{N}^I$, where $\bullet$ is the algebraic group action (3.11).

(b) *The map $X \mapsto f_X$* (3.6) *together with the isomorphisms in* (a) *defines a bijection between the isomorphism classes of modules X in* $\mathrm{prin}\, K(I,\mathfrak{Z})$ *with* $\mathbf{cdn}(X) = v$ *and the* $\mathbf{G}_v^{(I,\mathfrak{Z})}$*-orbits in* $\mathbf{Mat}_v^{(I,\mathfrak{Z})}$.

(c) *The category* $(I, \mathfrak{Z})$-spr *of peak* $(I, \mathfrak{Z})$-*spaces is of finite representation type if and only if the category* $\operatorname{prin} K(I, \mathfrak{Z})$ *is of finite representation type.*

(d) *If* $(I, \mathfrak{Z})$-spr *of peak* $(I, \mathfrak{Z})$-*spaces is of finite representation type then the following statements hold*:

(i) *For every coordinate vector* $v \in \mathbb{N}^I$ *the algebraic* K-*variety* $\mathbf{Mat}_v^{(I,\mathfrak{Z})}$ *has finitely many* $\mathbf{G}_v^{(I,\mathfrak{Z})}$-*orbits with respect to the action* $\bullet$ *above.*

(ii) *The Tits quadratic form* $q_{(I,\mathfrak{Z})}$ (2.10) *is weakly positive.*

(iii) *The poset* $(I, \mathfrak{Z})$ *with zero-relations does not contain as a peak subposet any of the posets with zero-relations presented by Weichert in* [33, pp. 103–120].

Proof. The statements (a) and (b) follow from the discussion above, whereas (c) is a consequence of the properties of the adjustment functors $\boldsymbol{\Theta}$ (3.2) and $\boldsymbol{\rho}$ (2.7) described earlier.

(d) The statements (i) and (ii) follow by standard arguments applied in the proof of Theorem 1.3 in [26]. For the proof of (iii) one shows that for each of the posets $(I, \mathfrak{Z})$ with zero-relations from the list of Weichert [33, pp. 103–120] there exists a non-zero vector $\boldsymbol{\mu}_{(I,\mathfrak{Z})} = (\mu_i)_{i \in I} \in \mathbb{N}^I$ such that $q_{(I,\mathfrak{Z})}(\boldsymbol{\mu}_{(I,\mathfrak{Z})}) = 0$ and $\mu_j = 1$ for some $j \in I$. In all cases the set of zero-relations $\mathfrak{Z}$ is empty the vectors $\boldsymbol{\mu}_{(I,\mathfrak{Z})}$ are presented in [11, Section 5] and [26, Section 5]. In the remaining cases the proof is left to the reader. Then (iii) is a consequence of (ii). □

Remark 3.13. (a) One can prove that for every poset $(I, \mathfrak{Z})$ with zero-relations presented by Weichert in [33, pp.103–120; 34] the subset

$$\operatorname{Ker} q_{(I,\mathfrak{Z})} = \{v \in \mathbb{Z}^I;\ q_{(I,\mathfrak{Z})}(v) = 0\}$$

of $\mathbb{Z}^I$ is an infinite cyclic subgroup of $\mathbb{Z}^I$ generated by the vector $\boldsymbol{\mu}_{(I,\mathfrak{Z})}$ indicated in the proof above.

(b) One can prove that in Proposition 3.12 the following implications hold (d)⇒(i)⇒(ii)⇒(iii). There is an open problem, when the condition (iii) implies that the category $(I, \mathfrak{Z})$-spr is of finite representation type? This is the case if the set $\mathfrak{Z}$ is empty [26] (see also [10]).

4. A reduction functor from lattices to socle projective representations

We shall study the category $\operatorname{latt}(\Lambda^\bullet)$ of lattices over the D-order $\Lambda^\bullet$ (1.3) by applying the Tits quadratic form (1.4) and the reduction functor

$$(4.1) \qquad \mathbb{H} : \operatorname{latt}(\Lambda^\bullet) \to (I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \cong \operatorname{mod}_{\mathrm{sp}} K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$$

defined below, where $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ is a two-peak poset with zero-relations associated with $\Lambda^\bullet$. The construction of $\mathbb{H}$ and $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ follows that one in [28, Section 2] and involves the covering technique for bipartite stratified posets developed by the author in [24].

Suppose that Λ, Λ_1, Λ_2 and Λ_3 are tiled D-orders in (1.2). In order to define $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ we consider the poset $(I_\Lambda; \preceq)$ (see [37]), where

(4.2) $\quad I_\Lambda = \{1,\dots,n\} \quad \text{and} \quad i \prec j \Leftrightarrow {}_iD_j = D.$

First we associate with the D-order $\Lambda^\bullet$ (1.3) the following combinatorial object

(4.3) $\quad I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'')$

where $(I_\Lambda, \preceq)$ is the poset (4.2), $I' = I_{\Lambda_1} = \{1,2,\dots,n_1\}$, $I'' = I_{\Lambda_2} = \{n_1+n_3+1,\dots, n-1,n\}$ and $C = I_{\Lambda_3} = \{n_1+1 \prec \cdots \prec n_1+n_3-1 \prec n_1+n_3\}$ are viewed as subposets of I_Λ such that $I_\Lambda = I' \cup C \cup I''$ is a splitting decomposition of I_Λ in the sense of [25, Section 8.1], and $\sigma : I' \to I''$ is the poset isomorphism defined by the formula $\sigma(j) = n_1 + n_3 + j$. It is clear that $I_{\Lambda^\bullet,\sigma}$ is a bipartite stratified poset in the sense of [24] and [25, Section 17.8], or a completed poset in the sense of [14].

Let $C' = \{c'; c \in C\}$ be a chain isomorphic with C. We construct two one-peak enlargements

$$(C \cup I'')^* = C \cup I'' \cup \{*\} \quad \text{and} \quad (I' \cup C)^+ = I' \cup C' \cup \{+\}$$

of the posets $C \cup I''$ and $I' \cup C \equiv I' \cup C'$ by the unique maximal points $*$ and $+$, and by the new relations $i \prec *$ and $s \prec +$ for all $i \in C \cup I''$ and all $s \in I' \cup C'$.

Next we associate with the D-order $\Lambda^\bullet$ (1.3) the two-peak poset

(4.4) $\quad I_{\Lambda^\bullet}^{*+} = (C \cup I'')^* \bigcup_{I'' \equiv I'} (I' \cup C)^+$

obtained from the disjoint union $(C \cup I'')^* \cup (I' \cup C)^+$ of $(C \cup I'')^*$ and $(I' \cup C)^+$ by making the identification $j \equiv \sigma(j)$ for any element $j \in I' \subseteq (I' \cup C)^+$.

It is easy to see that $I_{\Lambda^\bullet}^{*+}$ is a poset and $\max I_{\Lambda^\bullet}^{*+} = \{*,+\}$. The reader is referred to [28, 2.5] for illustrating examples.

We define the set $\mathfrak{Z}_{\Lambda^\bullet}$ of zero-relations in $I_{\Lambda^\bullet}^{*+}$ to be the set of all such a pairs (c, c_1') that $c \in C \subseteq (C \cup I'')^*$, $c_1' \in C' \subseteq (I' \cup C)^+$ and the relations $c \prec s$, $\sigma(s) \prec c_1$ hold in I_Λ for some $s \in I'$. Here we use the convention $+' = +$.

We call the pair $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ a *poset with zero-relations associated with the D-order $\Lambda^\bullet$*.

Example 4.5. Let $\Gamma_\bullet$ be the subamalgam of the three-partite D-order

$$\Gamma = \left(\begin{array}{ccc|ccc|ccc}
D & D & D & p & p & D & D & D & D \\
p & D & D & p & p & D & D & D & D \\
p & p & D & p & p & D & D & D & D \\
\hline
p & p & p & D & D & D & D & D & D \\
p & p & p & p & D & D & D & D & D \\
p & p & p & p & p & D & p & p & D \\
\hline
p & p & p & p & p & p & D & D & D \\
p & p & p & p & p & p & p & D & D \\
p & p & p & p & p & p & p & p & D
\end{array}\right) \quad n_3 = 3$$

in the sense of definition (1.3). Then $n_1 = n_2 = n_3 = 3$, the Tits quadratic form $q_{\Gamma^\bullet} : \mathbb{Z}^{11} \to \mathbb{Z}$ is the quadratic form

$$\begin{aligned}
&q_{\Gamma^\bullet}(x_1,x_2,x_3,x_4,x_5,x_6,\bar{x}_4,\bar{x}_5,\bar{x}_6,x_*,x_+)\\
&\quad = x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+\bar{x}_4^2+\bar{x}_5^2+\bar{x}_6^2+x_*^2+x_+^2\\
&\qquad + x_1x_2+x_1x_3+x_2x_3+(x_1+x_2+x_3)x_6+x_4x_5+x_4x_6+x_5x_6\\
&\qquad + \bar{x}_4\bar{x}_5+\bar{x}_4\bar{x}_6+\bar{x}_5\bar{x}_6+(x_1+x_2+x_3)(\bar{x}_4+\bar{x}_5)+x_3\bar{x}_6\\
&\qquad - x_+(x_1+x_2+x_3+x_4+x_5+x_6)-x_*(x_1+x_2+x_3+\bar{x}_4+\bar{x}_5+\bar{x}_6).
\end{aligned}$$

The poset $(I_\Gamma, \preceq)$ of Γ in the sense of (4.2) is the following:

$$I_\Gamma : \quad \begin{tikzcd}
 & 4 \arrow[r] & 5 \arrow[r] \arrow[dr] & 6 & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[ur] & 7 \arrow[r] & 8 \arrow[r] & 9
\end{tikzcd}$$

$I_\Gamma = I' \cup C \cup I''$ is a splitting decomposition of I_Γ, where $I' = \{1 \to 2 \to 3\}$, $C = \{4 \to 5 \to 6\}$, $I'' = \{7 \to 8 \to 9\}$. The poset isomorphism $\sigma : I' \to I''$ is the obvious one (see (4.3)). Note that

$$(C \cup I'')^* : \quad \begin{tikzcd}
4 \arrow[r] \arrow[dr] & 5 \arrow[r] & 6 \arrow[dr] & & \\
 & 7 \arrow[r] & 8 \arrow[r] & 9 \arrow[r] & *
\end{tikzcd}$$

$$(I' \cup C)^+ : \quad \begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[dr] & & \\
 & 4' \arrow[r] & 5' \arrow[r] & 6' \arrow[r] & +
\end{tikzcd}$$

$$(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet}) : \quad \begin{tikzcd}
4 \arrow[r] \arrow[d] & 5 \arrow[r] & 6 \arrow[dr] \arrow[ddrr, dashed] & & & \\
1 \arrow[r] & 2 \arrow[rr] & & 3 \arrow[r] \arrow[dr] & * & \\
 & & 4' \arrow[r] & 5' \arrow[r] & 6' \arrow[r] & +
\end{tikzcd}$$

The set $\mathfrak{Z}_{\Gamma^\bullet}$ consists of six zero-relations $(6,6')$, $(5,6')$, $(4,6')$, $(6,+)$, $(5,+)$, $(4,+)$. It is generated by the unique minimal relation $(6,6')$. This defines a poset $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$ with zero-relations.

It is easy to see that $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$ does not contain as a two-peak subposet with zero-relations any of the eight forms $\mathscr{F}_1, \ldots, \mathscr{F}_8$ presented in Theorem 1.6(c), and according to Theorem 1.6 the D-order $\Gamma^\bullet$ is of finite lattice type.

One can show that the set $\mathscr{R}^+_{\Lambda^\bullet}$ (1.9) of positive roots of the Tits quadratic form $q_{\Gamma^\bullet}$ of $\Gamma^\bullet$ has 110 elements (see Example 6.6 below). It then follows from Theorem 6.1 (c) of Section 6 that the number of the isomorphism classes of indecomposable $\Gamma^\bullet$-lattices is equal to 96. The Auslander-Reiten quiver of $\mathrm{latt}(\Gamma^\bullet)$ is described in Example 6.6 of Section 6.

(4.6) A construction of the functor $\mathbb{H}$. First we note that the order $\Lambda^\bullet$ (1.3) is a D-suborder of the hereditary order $\Gamma = \mathbb{M}_n(D)$ and the two-sided ideal $\pi = \mathbb{M}_n(\mathfrak{p})$ in the Jacobson radical $\mathrm{rad}(\Gamma)$ of Γ is also an ideal of $\Lambda^\bullet$ contained in $\mathrm{rad}(\Lambda^\bullet)$. It is not difficult to check that the ring

$$R = \begin{pmatrix} \Lambda^\bullet/\pi & \Gamma/\pi \\ 0 & \Gamma/\pi \end{pmatrix}$$

is a finite-dimensional right peak K-algebra (see [25]) with a unique simple right ideal P_* up to isomorphism, and according to [8, 19] the reduction functor

(4.7) $\quad \mathbb{F} : \mathrm{latt}(\Lambda^\bullet) \to \mathrm{mod}_{\mathrm{sp}}\, R$

defined by the formula $\mathbb{F}(X) = (X/X\pi, X\Gamma/X\pi, u)$, is full, reflects isomorphisms and $\mathrm{Im}\,\mathbb{F}$ contains up to isomorphism all indecomposable objects of $\mathrm{mod}_{\mathrm{sp}}\, R$ except from P_*. Here $X\Gamma$ is the Γ-submodule of $X \otimes_D F$ generated by X (see [17]), $F = D_0$ is the field of fractions of D and $u : X/X\pi \to X\Gamma/X\pi$ is the Λ/π-monomorphism induced by the natural monomorphism $X \hookrightarrow X\Gamma$. We view $\mathbb{F}(X)$ as a right R-module in a natural way (see [8, 19]).

Let $J = I^*_\Lambda = I_\Lambda \cup \{*\}$ be the poset obtained from I_Λ by adding the unique maximal element $*$ with new relations $i \prec *$ for all $i \in I_\Lambda$. Consider the set

$$\blacktriangle J := \{(i,j);\ i \preceq j \text{ in } J\} \subseteq J \times J$$

and define a binary equivalence relation ρ on $\blacktriangle J$ by setting

$$(i,j)\rho(s,t) \;\Leftrightarrow\; (i,j) = (s,t) \quad \text{or} \quad i,s \in I' = I_{\Lambda_1}, j,t \in I'' = I_{\Lambda_2}, j = \sigma(i), t = \sigma(s)$$

where $\sigma : I' \to I''$ given by $\sigma(i) = i + n_1 + n_3$ is a poset isomorphism. Then we have defined a bipartite stratified poset

(4.8) $\quad J_\rho = (J, \rho)$

in the sense of [22, 24, Definition 4.1] and the bipartition $J = J' + C + J'''$ is given by taking $J' = I'$, $C = I_{\Lambda_3}$ and $J''' = (I'')^*$. We recall from [22, 24] that the incidence K-algebra of J_ρ is the subalgebra KJ_ρ of KJ consisting of all matrices $\lambda = [\lambda_{pq}]_{p,q \in J}$ such that $\lambda_{ij} = \lambda_{st}$ if $(i,j),(s,t) \in \blacktriangle J$ and $(i,j)\rho(s,t)$. It was shown in [24] that KJ_ρ is a basic right peak K-algebra and the right socle of KJ_ρ is isomorphic to a direct sum of the simple projective right ideal $P'_* = e_*KJ_\rho$, called a right peak of KJ_ρ. A simple analysis shows that the algebra R defined above is Morita equivalent with the incidence algebra KJ_ρ and therefore there exists an equivalence of categories

(4.9) $\qquad G : \mathrm{mod}_{\mathrm{sp}} R \xrightarrow{\simeq} \mathrm{mod}_{\mathrm{sp}}\, KJ_\rho$

preserving finite representation type, tameness, wildness and the polynomial growth property. A simple illustration of these facts can be found in [29, p.96].

Let $(Q, \Omega) = (Q(J_\rho), \Omega(J_\rho))$ be the bound quiver associated with J_ρ in [24, Definition 2.5]. It follows from [24, Proposition 2.8] that there exists a K-algebra isomorphism $K(Q, \Omega) \cong KJ_\rho$. Let

$$f : (\widetilde{Q}, \widetilde{\Omega}) \to (Q, \Omega)$$

be the bound quiver Galois covering [24, (3.1)] of (Q, Ω). It follows from [24, Proposition 3.8] that f is a universal covering with the covering group $\mathbb{Z}$. Moreover, it follows from the construction that the two-peak bound quiver J_ρ^{*+} [24, (4.3)] of J_ρ is just the poset $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations (4.4) associated with $\Lambda^\bullet$. By [24, Theorem 4.19] the push-down functor $f_\lambda : \mathrm{Mod}\, K(\widetilde{Q}, \widetilde{\Omega}) \to \mathrm{Mod}\, K(Q, \Omega)$ induces the push-down functor

$$\mathrm{mod}_{\mathrm{sp}}\, K(\widetilde{Q}, \widetilde{\Omega}) \xrightarrow{f_{sp}} \mathrm{mod}_{\mathrm{sp}}\, K(Q, \Omega) \cong \mathrm{mod}_{\mathrm{sp}}\, KJ_\rho$$

and we get the following diagram

$$\begin{tikzcd}
\mathrm{mod}_{\mathrm{sp}} K(Q,\Omega) & \mathrm{mod}_{\mathrm{sp}} K(\widetilde{Q},\widetilde{\Omega}) \arrow[l, "f_{sp}"'] \\
\mathrm{mod}_{\mathrm{sp}}\, KJ_\rho \arrow[u, "\cong"] \arrow[r, shift right, "f^-"'] & \mathrm{mod}_{\mathrm{sp}}\, KJ_\rho^{*+} \arrow[u, "\Phi"'] \arrow[l, shift right, "f^+"'] \\
\mathrm{latt}(\Lambda^\bullet) \arrow[u, "G \circ \mathbb{F}"'] \arrow[r, "\mathbb{H}"] & (I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \arrow[u, "\cong", "T"']
\end{tikzcd} \tag{4.10}$$

where f^+ is the glueing functor [24, (4.14)], $\Phi = T_v \circ L_\xi$ is the embedding defined in [24, Proposition 4.23], f^- is the section functor [24, (5.1)], and in view of the identification $J_\rho^{*+} \equiv (I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ the functor T is the equivalence of categories ρ : $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr $\to \mathrm{mod}_{\mathrm{sp}}\, K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ defined in (2.7). The above facts are explained by Example 3.9 in [29, p.96].

Definition 4.11. The reduction functor $\mathbb{H} : \mathrm{latt}(\Lambda) \to (I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr $\cong \mathrm{mod}_{\mathrm{sp}}\ K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ (4.1) is the composed functor $\mathbb{H} = T^{-1} \circ f^- \circ G \circ \mathbb{F}$.

We shall use two chains of representations in $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr

$$\begin{aligned}
&\mathbf{P}_{n_3}^+ \hookrightarrow \mathbf{P}_{n_3-1}^+ \hookrightarrow \cdots \hookrightarrow \mathbf{P}_0^+ \\
&\mathbf{H}_{n_3}^- \hookrightarrow \mathbf{H}_{n_3-1}^- \hookrightarrow \cdots \hookrightarrow \mathbf{H}_0^-
\end{aligned} \tag{4.12}$$

defined as follows. Given $c \in \{0,1,\dots,n_3\}$ we define $\mathbf{P}_c^+ = (P_{c,j}^+)_{j\in I_{\Lambda^\bullet}^{*+}}$ and $\mathbf{H}_c^- = (H_{c,j}^-)_{j\in I_{\Lambda^\bullet}^{*+}}$ by the formulas

$$P_{c,j}^+ = \begin{cases} K & \text{if } j \succeq (c+n_1)' \text{ in } C' \cup \{+\} \subseteq I_{\Lambda^\bullet}^{*+}, \\ 0 & \text{elsewhere,} \end{cases}$$

$$H_{c,j}^- = \begin{cases} K & \text{if } j \in I'' \equiv I', \text{ or } j \succ c+n_1 \text{ in } C \cup \{*\} \subseteq I_{\Lambda^\bullet}^{*+}, \\ 0 & \text{elsewhere.} \end{cases}$$

In particular, $H_{0,j}^- = K$ for all $j \in (C \cup I'')^* \subseteq I_{\Lambda^\bullet}^{*+}$, and $H_{0,j}^- = 0$ for all $j \in C' \cup \{+\} \subseteq I_{\Lambda^\bullet}^{*+}$. We identify any element $i \in I'$ with $\sigma(i) = i+n_1+n_3 \in I''$ in the poset $I_{\Lambda^\bullet}^{*+}$.

It follows that, under the identification $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr $\cong \operatorname{mod}_{\mathrm{sp}} K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$, the representation $\mathbf{H}_0^-$ is the injective envelope of the simple projective representation $e_* K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ corresponding to the peak $*$, and there are isomorphisms $\mathbf{P}_{n_3}^+ \cong e_+ K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ and $\mathbf{P}_c^+ \cong e_{(c+n_1)'} K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ for $c = 1,\dots,n_3$, where $(c+n_1)'$ is viewed as an element of $C' \subseteq I_{\Lambda^\bullet}^{*+}$.

It follows from [23, Lemma 3.8] that $\mathbf{P}_{n_3}^+ \hookrightarrow \mathbf{P}_{n_3-1}^+ \hookrightarrow \cdots \hookrightarrow \mathbf{P}_0^+$ are hereditary projective representations and $\mathbf{H}_{n_3}^- \hookrightarrow \mathbf{H}_{n_3-1}^- \hookrightarrow \cdots \hookrightarrow \mathbf{H}_0^-$ are hereditary sp-injective representations in $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr $\cong \operatorname{mod}_{\mathrm{sp}} K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$.

Let $P_1,\dots,P_{n_1+n_3}$ be a complete set of indecomposable projective right $\Lambda^\bullet$-modules. We recall from [27, Section 3; 30, 7.53] that every lattice Y in latt$(\Lambda^\bullet)$ admits a projective cover epimorphism $P(X) \to Y$ and $P(Y) \cong P_1^{w_1} \oplus \cdots \oplus P_{n_1+n_3}^{w_{n_1+n_3}}$ for some non-negative integers $w_1,\dots,w_{n_1+n_3} \in \mathbb{N}$. Following [27, Section 3] and [30, 7.53] we call

(4.13) $\quad \mathbf{cdn}(Y) = (w_1,\dots,w_{n_1+n_3})$

a *coordinate vector* of the lattice Y. We denote by $\mathrm{ind}_s(\mathrm{latt}(\Lambda^\bullet))$ and by $\mathrm{ind}_w(\mathrm{latt}(\Lambda^\bullet))$ the set of the isomorphism classes of indecomposable $\Lambda^\bullet$-lattices Y of D-rank s and with $\mathbf{cdn}(Y) = w$, respectively.

Now we are able to prove our main reduction theorem.

Theorem 4.14. *Assume that D is a complete discrete valuation domain, $\mathfrak{p}$ is the unique maximal ideal of D and $K = D/\mathfrak{p}$. Let Λ be the D-order* (1.1) *with the three-partition* (1.2) *and $\Lambda_1 = \Lambda_2 \subseteq \mathbb{M}_{n_1}(D)$, $\Lambda_3 \subseteq \mathbb{M}_{n_3}(D)$ and n_1, n_3 as in Section* 1. *Let $\Lambda^\bullet$ be the subamalgam D-order* (1.3) *and let $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ be the two-peak poset* (4.4) *with zero-relations associated with $\Lambda^\bullet$. Then the following statements hold*:

(a) *The Tits quadratic forms $q_{\Lambda^\bullet}$* (1.4) *and $q_{(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})}$* (2.10) *coincide.*

(b) *The reduction functor $\mathbb{H}$: latt$(\Lambda^\bullet) \to (I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr $\cong \operatorname{mod}_{\mathrm{sp}} K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$* (4.11) *is K-linear and has the following properties*:

(i) *$\mathbb{H}$ is full, reflects isomorphisms, preserves the indecomposability and preserves and reflects the finite representation type.*

(ii) *$\mathrm{Im}\,\mathbb{H}$ consists up to isomorphism of all objects of $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z})$-spr having no direct summand of one of the following two types: the simple projective representation $P_* = e_* K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z})$ corresponding to the peak idempotent e_* and any of* *the hereditary sp-injective representations* $\mathbf{H}_{n_3}^- \hookrightarrow \mathbf{H}_{n_3-1}^- \hookrightarrow \cdots \hookrightarrow \mathbf{H}_0^-$ *defined in* (4.12).

(iii) *The set* $\mathrm{ind}_s(\mathrm{latt}(\Lambda^\bullet))$ *is finite for any* $s \in \mathbb{N}$ *if and only if the set* $\mathrm{ind}_v((I_{\Lambda^\bullet}^{*+}, \mathfrak{Z})\text{-spr})$ *of the isomorphism classes of indecomposable objects* X *in* $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z})$-spr *with* $\mathbf{cdn}(X) = v$ *is finite for any vector* $v \in \mathbb{N}^{n_1+2n_3+2}$.

Proof. The statement (a) follows by a straightforward analysis. (b) First we shall prove (ii) by applying the properties of the functors $\mathbb{F} : \mathrm{latt}(\Lambda^\bullet) \to \mathrm{mod}_{\mathrm{sp}}\, R$ (4.7), $G : \mathrm{mod}_{\mathrm{sp}}\, R \xrightarrow{\simeq} \mathrm{mod}_{\mathrm{sp}} - KJ_\rho$ (4.9) defined above and by applying the properties of the functor $f^- : \mathrm{mod}_{\mathrm{sp}} - KJ_\rho \xrightarrow{\simeq} \mathrm{mod}_{\mathrm{sp}} - K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ (see (4.10)) given in [24, Theorem 5.8(b)].

For this purpose we set $R^+ = KJ_\rho^{*+}$ and recall that there is a bound quiver isomorphism $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet}) \cong J_\rho^{*+}$. Then there is an isomorphism $R^+ \cong K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ of K-algebras. We recall from the discussion above that the simple projective module $e_* KJ_\rho$ is not in the image of $G \circ \mathbb{F}$. It follows from the definition of f^- [24, (5.5)] that the functor f^- carries $e_* KJ_\rho$ to $P_* = e_* K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet}) \cong e_* R^+$ (see also [24, Theorems 4.27(a) and 5.8(b)].

It is easy to see that the composed functor

$$\mathrm{mod}_{\mathrm{sp}}\, KR^+ \xrightarrow{\simeq} \mathrm{mod}_{\mathrm{sp}}\, K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet}) \xrightarrow{\simeq} (I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}$$

carries the chain of modules $H_{n_3} \to \cdots \to H_0$ in $\mathrm{mod}_{\mathrm{sp}}\, KR^+$ defined in [24, (4.25)] (with m and n_3 interchanged) to the chain $\mathbf{H}_{n_3}^- \hookrightarrow \mathbf{H}_{n_3-1}^- \hookrightarrow \cdots \hookrightarrow \mathbf{H}_0^-$ of the hereditary sp-injective representations in $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z})$-spr defined in (4.12). It follows from [24, Theorem 5.8(b)] that the image of the composed functor $T^{-1} \circ f^-$ consists (up to isomorphism) of all objects of $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z})$-spr having no direct summand isomorphic to any of the hereditary sp-injective representations $\mathbf{H}_{n_3}^- \hookrightarrow \mathbf{H}_{n_3-1}^- \hookrightarrow \cdots \hookrightarrow \mathbf{H}_0^-$. Hence (ii) follows.

Since the functors T^{-1}, f^- and $\mathbb{F}$ are full and reflects isomorphisms (see [24, Theorem 5.8(b)]) then so is $\mathbb{H}$ and in view of (ii) the statement (i) of (b) follows.

(iii) By [27, Lemma 3.8] and [30, 7.12] the set $\mathrm{ind}_s(\mathrm{latt}(\Lambda^\bullet))$ is finite for any $s \in \mathbb{N}$ if and only if the set $\mathrm{ind}_w(\mathrm{latt}(\Lambda^\bullet)$ is finite for any vector $w \in \mathbb{N}^{n_1+n_3}$. By the proof of [30, Proposition 7.11] and by [30, 7.20] applied to $\mathbf{F}_1 = \mathbb{F} \circ \mathrm{res}$ the above holds if and only if the sets $\mathrm{ind}_u(\mathrm{mod}_{\mathrm{sp}}\, KJ_\rho) \equiv \mathrm{ind}_u(\mathrm{mod}_{\mathrm{sp}}\, K(Q, \Omega))$ are finite for any vector $u \in \mathbb{N}^{n_1+n_3+1}$. It follows from [24, Corollary 4.29] that this is the case if and only if the sets $\mathrm{ind}_v(\mathrm{mod}_{\mathrm{sp}} KJ_\rho^{*+}) \equiv \mathrm{ind}_v((I_{\Lambda^\bullet}^{*+}, \mathfrak{Z})\text{-spr})$ are finite for any vector $v \in \mathbb{N}^{n_1+2n_3+2}$. This finishes the proof of the theorem. □

5. Proof of Theorem 1.6

Assume that Λ, Λ_1, Λ_2, Λ_3 and $\Lambda^\bullet$ are D-orders as in Theorem 1.6. We recall from the discussion in (4.6) that we have associated with $\Lambda^\bullet$ a bipartite stratified poset $J_\rho = (J, \rho)$ (4.8) and a bound quiver J_ρ^{*+}. It is easy to see that $J_\rho = (J, \rho)$ is just the bipartite stratified poset $I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'')$ (4.3) and J_ρ^{*+} is the poset $(I_\Lambda^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations (see (4.4)). Then there are equivalences of categories $(I_\Lambda^{*+}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \cong \mathrm{mod}_{\mathrm{sp}}\, K(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet}) \cong \mathrm{mod}_{\mathrm{sp}}\, KJ_\rho^{*+}$.

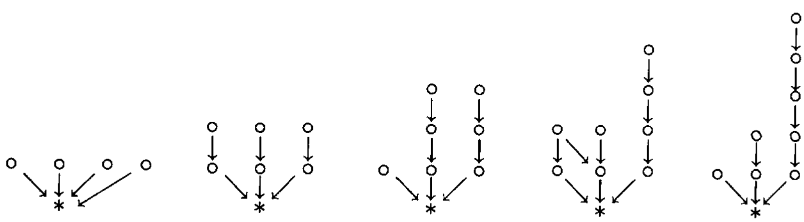
It follows from Theorem 4.14 that $\Lambda^\bullet$ is of finite lattice type if and only if the category $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr is of finite representation type. Moreover, the Tits quadratic forms $q_{\Lambda^\bullet}$ (1.4) and $q_{(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})}$ (2.10) coincide.

This together with [24, Theorems 4.30] shows that the following three statements are equivalent:

(a′) The category $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$-spr is of finite representation type.

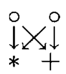
(b′) The Tits quadratic form $q_{(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})}$ is weakly positive.

(c′) The two-peak poset $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet}) \cong J_\rho^{*+}$ with zero-relations associated with $\Lambda^\bullet$ in (4.4) does not contain as a two-peak subposet any of the 48 two-peak posets with zero-relations presented in [24, Table 1, p. 3570], and does not contain as a peak subposet any of the following one-peak enlargements $\mathscr{K}_1^*$, $\mathscr{K}_2^*$, $\mathscr{K}_3^*$, $\mathscr{K}_4^*$, $\mathscr{K}_5^*$:



of critical Kleiner's posets (see [25, Theorem 10.1]).

It follows from the construction $\Lambda^\bullet \mapsto (I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ in (4.4) and from our assumption on Λ in (1.2) that the poset $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet}) \cong J_\rho^{*+}$ with zero-relations does not contain the poset



(being the first poset in the list of the 48 forms in [24, Table 1, p. 3570]) if and only if the poset $I' = I_{\Lambda_1}$ is linearly ordered, or equivalently, the D-order Λ_1 in (1.2) is hereditary of the form (1.7).

Note also that the poset $I_{\Lambda^\bullet}^{*+} \setminus (I' \cup \{*, +\})$ is a disjoint union of two chains C and C'. Then a case by case inspection of the 48 forms in [24, Table 1, p. 3570] shows that for any three-partite subamalgam D-order $\Lambda^\bullet$ (1.3) with Λ_1 of the form (1.7) the poset $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations does not contain as a peak subposet any of the one-peak enlargements $\mathscr{K}_1^*$, $\mathscr{K}_2^*$, $\mathscr{K}_3^*$, $\mathscr{K}_4^*$, $\mathscr{K}_5^*$ of critical Kleiner's posets, and $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ could contain at most the eight forms $\widetilde{\mathbb{E}}_6^2$, $\widetilde{\mathbb{E}}_7^5$, $\widetilde{\mathbb{E}}_7^4$, $\widetilde{\mathbb{E}}_8^{21}$, $\widetilde{\mathbb{E}}_8^{20}$, $\widetilde{\mathbb{E}}_8^{30}$, $\widetilde{\mathbb{E}}_8^{31}$ and $\widetilde{\mathbb{E}}_8^{35}$ from the 48 forms presented in [24, Table 1, p. 3570]. They are just the forms $\mathscr{F}_1, \ldots, \mathscr{F}_8$ shown in Theorem 1.6(c). It then follows that the condition (c) of Theorem 1.6 is equivalent with (c′) above. The remarks above prove also the equivalences (a)⇔(a′) and (b)⇔(b′). Consequently, the statements (a), (b) and (c) of Theorem 1.6 are equivalent.

Now we prove the implication (c)⇒(d). First we note that if the three-partite order $\Lambda^\bullet$ contains a three-partite minor D-suborder $\Gamma^\bullet$ then the two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations associated with $\Lambda^\bullet$ in (4.4) contains $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$ as a two-peak subposet with zero-relations.

Now for the proof of (c)⇒(d) it is sufficient to check (by applying the definition) that if $\Lambda^\bullet$ is any of the 14 D-orders $\Gamma_1^\bullet, \ldots, \Gamma_{14}^\bullet$ in Table 1 of Section 1 then $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains one of the forms $\mathscr{F}_1, \ldots, \mathscr{F}_8$ shown in Theorem 1.6(c) (see the proof of (d′)⇒(c′) in [31, Section 4] for details). We shall show this only in case $\Lambda^\bullet$ is any of the D-orders $\Gamma_1^\bullet$, $\Gamma_2^\bullet$, $\Gamma_3^\bullet$, $\Gamma_4^\bullet$ and $\Gamma_5^\bullet$. The proof in remaining cases is analogous and we leave it to the reader.

First we note that the sets $\mathfrak{Z}_{\Gamma_1^\bullet}$, $\mathfrak{Z}_{\Gamma_3^\bullet}$ and $\mathfrak{Z}_{\Gamma_4^\bullet}$ are empty, the sets $\mathfrak{Z}_{\Gamma_2^\bullet}$ and $\mathfrak{Z}_{\Gamma_5^\bullet}$ are not empty, $I^{*+}_{\Gamma_1^\bullet} = \mathscr{F}_1$ and $I^{*+}_{\Gamma_3^\bullet} = \mathscr{F}_2$.

The poset $I^{*+}_{\Gamma_2^\bullet}$ has the form

$$I^{*+}_{\Gamma_2^\bullet}: \quad \begin{tikzcd} & c_2 \arrow[r] \arrow[dl] & c_1 \arrow[r] & * \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[ur] \arrow[d] & \\ & c_2' \arrow[r] & c_1' \arrow[r] & + \end{tikzcd}$$

and $\mathfrak{Z}_{\Gamma_2^\bullet} = \{(c_2, c_1'), (c_2, +)\}$. It follows that the poset with zero-relations $(I^{*+}_{\Gamma_2^\bullet}, \mathfrak{Z}_{\Gamma_2^\bullet})$ contains the poset $\mathscr{F}_2$ as the subposet obtained from $I^{*+}_{\Gamma_1^\bullet}$ by omitting the points c_2 and c_1'.

The poset $I^{*+}_{\Gamma_4^\bullet}$ has the form

$$I^{*+}_{\Gamma_4^\bullet}: \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \circ \arrow[r] & * \\ \bullet \arrow[r] & \bullet \arrow[ur] \arrow[dr] & & \\ \bullet \arrow[r] & \bullet \arrow[r] & + & \end{tikzcd}$$

and therefore it contains the poset $\mathscr{F}_2$ as the subposet consisting of the solid points and the maximal points $*$, $+$.

The poset $I^{*+}_{\Gamma_5^\bullet}$ has the form

$$I^{*+}_{\Gamma_5^\bullet}: \quad \begin{tikzcd} a \arrow[r] & b \arrow[r] \arrow[dr] & \circ \arrow[r] & \circ \arrow[r] & * & & \\ & & \circ \arrow[r] & \circ \arrow[ur] \arrow[dr] & & & \\ & & \circ \arrow[r] & \circ \arrow[r] & c \arrow[r] & d \arrow[r] & + \end{tikzcd}$$

and $\mathfrak{Z}_{\Gamma_5^\bullet} = \{(a,c), (a,d), (a,+), (b,c), (b,c), (b,+)\}$. It follows that the poset with zero-relations $(I^{*+}_{\Gamma_5^\bullet}, \mathfrak{Z}_{\Gamma_5^\bullet})$ contains the poset $\mathscr{F}_2$ as the subposet obtained from $I^{*+}_{\Gamma_5^\bullet}$ by omitting the solid points.

The proof of the implication (d)⇒(c) reduces to pure combinatorial poset properties by applying the constructions

$$\Lambda^\bullet \mapsto I_{\Lambda^\bullet,\sigma} \mapsto (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$$

where $I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'')$ is the bipartite stratified poset (4.3) and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ is the two-peak poset with zero-relations (4.4).

The following properties of the constructions follow directly from definitions.

(A) The D-order Λ together with its three-partition shown in (1.2) is uniquely determined by the bipartite stratified poset $I_{\Lambda^\bullet,\sigma}$. Hence, the three-partite subamalgam D-order $\Lambda^\bullet$ (1.4) is uniquely determined by $I_{\Lambda^\bullet,\sigma}$.

(B) A three-partite subamalgam D-order $\Gamma^\bullet$ is a three-partite minor D-suborder of $\Lambda^\bullet$ if and only if $I_{\Gamma^\bullet,\tau}$ is a bipartite stratified subposet of $I_{\Lambda^\bullet,\sigma}$.

(C) For any bipartite stratified subposet $J_\tau = (J, \preceq, J', C, J'', \tau : J' \to J'')$ of $I_{\Lambda^\bullet,\sigma}$ there exists a unique three-partite minor D-suborder Γ of Λ such that $I_{\Gamma^\bullet,\tau} = J_\tau$.

(D) A three-partite D-order Λ' of the form (1.2) dominates a three-partite D-order Λ if and only if $I' = I_{\Lambda_1} = I_{\Lambda'_1}$, $I'' = I_{\Lambda_2} = I_{\Lambda'_2}$, $C = I_{\Lambda_3} = I_{\Lambda'_3}$ (a poset equality) and the partial order relation of $I_{\Lambda'}$ is obtained from the partial order relation of I_Λ by adding finitely many new relations $i' \preceq c_1$, $c_2 \preceq i''$, where $i' \in I'$, $i'' \in I''$ and $c_1, c_2 \in C$.

(E) If the two-peak poset with zero-relations $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ is given then the poset $I' \equiv I''$ can be reconstructed as the subposet of $I^{*+}_{\Lambda^\bullet}$ consisting of all points s such that $s \preceq *$, $s \preceq +$ and each of the pairs $(s,*)$ and $(s,+)$ does not belong to the set $\mathfrak{Z}_{\Lambda^\bullet}$ of zero-relations. Moreover, $C \cup C' = I^{*+}_{\Lambda^\bullet} \setminus (I' \equiv I'')$ in the notation of (4.4).

It follows that the classification of minimal three-partite D-orders of infinite lattice type can be given by means of bipartite stratified subposets.

On this way we shall show that if Λ is a three-partite D-order (1.2) and the associated two-peak poset with zero-relations $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains one of the forms $\mathscr{F}_1, \ldots, \mathscr{F}_8$ as a two-peak subposet with zero-relations then the subamalgam D-order $\Lambda^\bullet$ (1.3) contains a three-partite minor D-suborder $\Gamma^\bullet$ which is dominated by any of the D-orders $\Gamma^\bullet_1, \ldots, \Gamma^\bullet_{14}$ shown in Table 1 of Section 1.

First we assume that Λ is a three-partite D-order (1.2) such that $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains the poset

$$\mathscr{F}_1:\quad \begin{tikzcd}[row sep=small] & & c_1 \arrow[r] & * \\ a_3 \arrow[r] & a_2 \arrow[r] & a_1 \arrow[ur] \arrow[dr] & \\ & & c_2' \arrow[r] & + \end{tikzcd}$$

and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ does not contain the poset $\mathscr{F}_2$. We shall show that the subamalgam D-order $\Lambda^\bullet$ contains a three-partite minor D-suborder $\Gamma^\bullet$ which is dominated by the D-order $\Gamma^\bullet_1$ or by $\Gamma^\bullet_2$ shown in Table 1.

Look at the bipartite stratified poset $I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'')$ (4.3). Recall that C is a chain, the elements c_1, c_2 belongs to C and c_2' denotes a copy of c_2 in

$C' \subseteq I^{*+}_{\Lambda\bullet}$ (see (4.4)). Without loss of generality we may suppose that $a_3 \preceq a_2 \preceq a_1$ is a chain in I' and $a_3' \preceq a_2' \preceq a_1'$ is the image of $a_3 \preceq a_2 \preceq a_1$ under the poset isomorphism $\sigma : I' \to I''$. It follows from our assumption on the bipartition (1.2) that $a_1 \preceq a_3'$.

Let Γ be a three-partite minor of Λ (1.2) defined by the rows and columns numbered by the elements $a_3, a_2, a_1, a_3', a_2', a_1', c_1, c_2$. By our assumption

$$I_{\Gamma\bullet,\sigma} = (J_\Gamma, \preceq, J', \overline{C}, J'', \sigma : J' \to J'')$$

where $J' = \{a_3 \preceq a_2 \preceq a_1\} \subset I'$, $J'' = \{a_3' \preceq a_2' \preceq a_1'\} \subset I''$, $\overline{C} = \{c_1, c_2\} \subseteq C$ and $\sigma : J' \to J''$ is given by $\sigma(a_1) = a_1'$, $\sigma(a_2) = a_2'$ and $\sigma(a_3) = a_3'$.

It follows from the shape of $\mathscr{F}_1$ that c_1 is not comparable with the chain $a_3' \to a_2' \to a_1'$ in the poset I_Λ and c_2 is not comparable with the chain $a_3 \to a_2 \to a_1$, and either $c_1 = c_2$ or else $c_2 \prec c_1$.

In case $c_1 = c_2$ we conclude from (A)–(C) and from the shape of the bipartite stratified poset $I_{\Gamma\bullet,\sigma}$ that $\Gamma = \Gamma_1$.

Now consider the case $c_2 \prec c_1$. Since $(I^{*+}_{\Gamma\bullet}, \mathfrak{Z}_{\Gamma\bullet})$ does not contains the poset $\mathscr{F}_2$ it follows from the observations above together with (A)–(E) that the poset $J_\Gamma = J' \cup \overline{C} \cup J''$ admits one of the following shapes:

$$J_1:\quad \begin{tikzcd} J_1': & a_3 \arrow[r] & a_2 \arrow[r] & a_1 \arrow[d] \arrow[ddll] \\ \overline{C}_1: & c_2 \arrow[rr] \arrow[d] & & c_1 \\ J_1'': & a_3' \arrow[r] & a_2' \arrow[r] & a_1' \end{tikzcd}
\qquad
J_2:\quad \begin{tikzcd} J_2': & a_3 \arrow[r] & a_2 \arrow[r] & a_1 \arrow[d] \arrow[ddll] \\ \overline{C}_2: & c_2 \arrow[rr] \arrow[dr] & & c_1 \\ J_2'': & a_3' \arrow[r] & a_2' \arrow[r] & a_1' \end{tikzcd}$$

$$J_3:\quad \begin{tikzcd} J_3': & a_3 \arrow[r] & a_2 \arrow[r] \arrow[dr] & a_1 \arrow[ddll] \\ \overline{C}_3: & c_2 \arrow[rr] \arrow[d] & & c_1 \\ J_3'': & a_3' \arrow[r] & a_2' \arrow[r] & a_1' \end{tikzcd}
\qquad
J_4:\quad \begin{tikzcd} J_4': & a_3 \arrow[r] & a_2 \arrow[r] \arrow[dr] & a_1 \arrow[ddll] \\ \overline{C}_4: & c_2 \arrow[rr] \arrow[dr] & & c_1 \\ J_4'': & a_3' \arrow[r] & a_2' \arrow[r] & a_1' \end{tikzcd}$$

$$J_5:\quad \begin{tikzcd} J_5': & a_3 \arrow[r] \arrow[drr] & a_2 \arrow[r] & a_1 \arrow[ddll] \\ \overline{C}_5: & c_2 \arrow[rr] \arrow[dr] & & c_1 \\ J_5'': & a_3' \arrow[r] & a_2' \arrow[r] & a_1' \end{tikzcd}
\qquad
J_6:\quad \begin{tikzcd} J_4': & a_3 \arrow[r] & a_2 \arrow[r] \arrow[dr] & a_1 \arrow[ddll] \\ \overline{C}_6: & c_2 \arrow[rr] \arrow[drr] & & c_1 \\ J_6'': & a_3' \arrow[r] & a_2' \arrow[r] & a_1' \end{tikzcd}$$

$$J_7:\quad \begin{tikzcd} J_7': & a_3 \arrow[r] \arrow[drr] & a_2 \arrow[r] & a_1 \arrow[ddll] \\ \overline{C}_7: & c_2 \arrow[rr] \arrow[drr] & & c_1 \\ J_7'': & a_3' \arrow[r] & a_2' \arrow[r] & a_1' \end{tikzcd}$$

It is easy to see that for $s = 1,\dots,7$ the two-peak poset with zero-relations J_s^{*+} associated with J_s by applying the formula (4.4) contains $\mathscr{F}_1$. Further, it follows from **(D)** that the three-partite D-order corresponding to J_1 dominates the three-partite D-orders corresponding to $J_2,\dots,J_7$. Since just the three-partite D-order Γ_2 corresponds to J_1 then from the analysis above we conclude that up to domination and minors the minimal three-partite D-orders (1.2) such that $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ contains the poset $\mathscr{F}_1$ and $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ does not contain the poset $\mathscr{F}_2$ are just the D-orders Γ_1 and Γ_2 of Table 1.

By the technique applied above we also prove the implication (d)$\Rightarrow$(c) in remaining cases. The details are left to the reader. This completes the proof of Theorem 1.6. □

Let $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ be the two-peak poset with zero-relations associated with $\Lambda^\bullet$ in (4.4). We associate with the D-order $\Lambda^\bullet$ the system

$$(5.1) \qquad * : G_{\Lambda^\bullet}(v) \times \mathscr{M}_{\Lambda^\bullet}(v) \longrightarrow \mathscr{M}_{\Lambda^\bullet}(v), \quad v \in \mathbb{N}^{n_1+2n_3+2},$$

of parabolic algebraic groups $G_{\Lambda^\bullet}(v) = \mathbf{G}_v^{(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})}$ acting on irreducible algebraic K-varieties $\mathscr{M}_{\Lambda^\bullet}(v) = \mathbf{Mat}_v^{(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})}$. The action is defined by the formula (3.11), with $(I, \mathfrak{Z})$ and $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ interchanged.

We summarise the results of Theorem 1.6 and of Section 3 by the following useful fact.

Corollary 5.2. *Assume that D is a complete discrete valuation domain. Let Λ be a three-partite D-order* (1.2) *and let $\Lambda^\bullet$ be the subamalgam* (1.3) *of Λ, where $\Lambda_1 = \Lambda_2$, Λ_3 and n, $n_1 = n_2$, n_3 are as in introduction. Let $(I_{\Lambda^\bullet}^{*+}, \mathfrak{Z}_{\Lambda^\bullet})$ be the two-peak poset with zero relations associated with $\Lambda^\bullet$ in* (4.4). *The following conditions are equivalent.*

(a) *The D-order $\Lambda^\bullet$ is of finite lattice type.*

(a′) *For any positive integer the set* $\mathrm{ind}_s(\mathrm{latt}(\Lambda^\bullet))$ *of the isomorphism classes of indecomposable $\Lambda^\bullet$- lattices of D-rank s is finite.*

(b) *The number of $G_{\Lambda^\bullet}(v)$-orbits in $\mathscr{M}_{\Lambda^\bullet}(v)$ with respect to the algebraic group action* (5.1) *is finite for all vectors $v \in \mathbb{N}^{n_1+2n_3+2}$.*

(c) $\dim\ G_{\Lambda^\bullet}(v) > \dim\ \mathscr{M}_{\Lambda^\bullet}(v)$ *for all $v \in \mathbb{N}^{n_1+2n_3+2}$, where* dim *is the variety dimension.*

(d) *The integral reduced Tits quadratic form $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \longrightarrow \mathbb{Z}$* (1.4) *is weakly positive.*

(e) *If $R = KI_{\Lambda^\bullet}^{*+}/\mathfrak{Z}_{\Lambda^\bullet}$ is the factor algebra of the incidence K-algebra $KI_{\Lambda^\bullet}^{*+}$ of the poset $I_{\Lambda^\bullet}^{*+}$ modulo the ideal $\mathfrak{Z}_{\Lambda^\bullet}$ of zero-relations then*

$$\dim_K\ \mathrm{Ext}_R^1(U_1, U_2) \le 1$$

for any pair of modules U_1, U_2 in $\mathrm{mod}(R)$ *satisfying the following conditions*:

(i) $\mathrm{End}(U_1) \cong \mathrm{End}(U_2) \cong K$,

(ii) $\mathrm{Hom}_R(U_1, U_2) = 0$ *and* $\mathrm{Hom}_R(U_2, U_1) = 0$,

(iii) *for $j = 1, 2$ the kernel of the projective cover $P(U_j) \to U_j$ of U_j is a semisimple projective R-module.*

(f) *The set* $\mathscr{R}^{+}_{\Lambda^\bullet} = \{v \in \mathbb{N}^{n_1+2n_3+2};\ q_{\Lambda^\bullet}(v) = 1\}$ (1.9) *of positive roots of the quadratic form* $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z}$ (1.4) *is finite.*

Proof. Since the functor $\mathbb{H} : \mathrm{latt}(\Lambda^\bullet) \to (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \cong \mathrm{mod}_{\mathrm{sp}} K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ (4.1) preserves and reflects the finite representation type then the condition (a) is equivalent to any of the following equivalent conditions:

(a1) The category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr is of finite representation type.

(a2) The category prin(R) of prinjective modules over $R = KI_{\Lambda^\bullet}/\mathfrak{Z}_{\Lambda^\bullet}$ is of finite representation type.

Moreover by Theorem 4.13 (a) the Tits quadratic form $q_{\Lambda^\bullet}$ (1.4) coincides with the Tits quadratic form $q_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ (2.10) of the two-peak poset with zero-relations $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$.

The implication (a)⇒(a′) is obvious.

(a′)⇒(b) It follows from Theorem 4.14(iii) that the condition (a′) implies the following one:

(a3) For any vector $v \in \mathbb{N}^{n_1+2n_3+2}$ the set $\mathrm{ind}_v(\mathrm{prin}\, R)$ of the isomorphism classes of indecomposable modules X in prin(R) with $v = \mathbf{cdn}(X)$ is finite.

By Proposition 3.12, the map $X \mapsto f_X$ (3.5) together with the isomorphisms in Proposition 3.12 (a) defines a bijection between the isomorphism classes of modules X in $\mathrm{prin}\, K(I, \mathfrak{Z})$ with $\mathbf{cdn}(X) = v$ and the $G_{\Lambda^\bullet}(v)$-orbits in $\mathscr{M}_{\Lambda^\bullet}(v)$. Then (b) follows by the arguments applied in the proof of Theorem 10.1 in [25, p. 128] or [26, Theorem 3.1].

The implications (b)⇒(c)⇒(d) can be proved by repeating the well-known Tits arguments used in the proof of Theorem 10.1 in [25, pp. 128–129] or [26, Theorem 3.1], because $q_{\Lambda^\bullet}$ coincides with $q_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ and obviously $q_{\Lambda^\bullet}(v) = \dim G_{\Lambda^\bullet}(v) - \dim \mathscr{M}_{\Lambda^\bullet}(v)$.

The implication (d)⇒(a) follows from Theorem 1.6. The equivalences (d)⇔(e)⇔(f) are proved in [32, Theorem 1.7] by applying the results and a technique of the papers [7] and [13]. This finishes the proof. □

We note that the equivalence (a)⇔(a′) is a version of the second Brauer-Thrall conjecture for lattices over orders (see [25, p. 4; 35]).

6. A construction of the Auslander–Reiten quiver of latt($\Lambda^\bullet$)

The main aim of this section is to present a construction of the Auslander–Reiten quiver of $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ of latt$(\Lambda^\bullet)$ in case $\Lambda^\bullet$ is of the form (1.3) with Λ_1 hereditary of the form (1.7). Theorem 6.1 below reduces the problem to a description of the Auslander–Reiten quiver of $\Gamma((I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr})$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$. This provides us with a computer accessible algorithm constructing the quiver $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ of latt$(\Lambda^\bullet)$ in case $\Lambda^\bullet$ is of finite lattice type (see Remark 6.4). The reader is referred to [4, 25] for basic facts on Auslander–Reiten sequences and Auslander–Reiten quivers (see also [15, 18, 35, Appendix; 36].

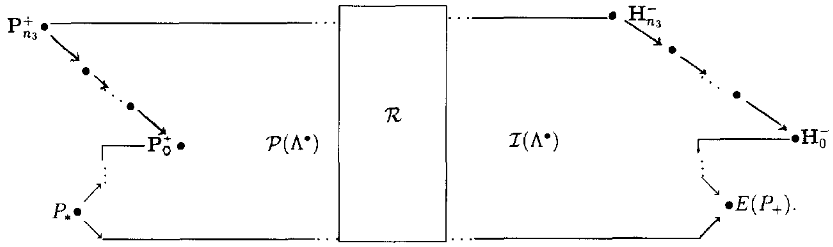


Fig. 1. Auslander–Reiten quiver of the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr.

Our main result of this section is the following.

Theorem 6.1. *Assume that D is a complete discrete valuation domain, $\mathfrak{p}$ is the unique maximal ideal of D and $K = D/\mathfrak{p}$.*

Let Λ be the D-order (1.1) *with the three-partition* (1.2) *and $\Lambda_1 = \Lambda_2 \subseteq \mathbb{M}_{n_1}(D)$, $\Lambda_3 \subseteq \mathbb{M}_{n_3}(D)$ and n_1, n_3 as in Section* 1. *Let $\Lambda^\bullet$ be the subamalgam D-order* (1.3) *and let $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ be the two-peak poset* (4.4) *with zero-relations associated with $\Lambda^\bullet$. If the order Λ_1 is hereditary of the form* (1.7) *then the following statements hold.*

(a) *There exist a unique preprojective component $\mathscr{P}(\Lambda^\bullet)$ in $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$*-spr *starting from the hereditary projective section $\mathbf{P}^+_{n_3} \hookrightarrow \mathbf{P}^+_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{P}^+_0$ of irreducible monomorphisms shown in* (4.12), *and a unique preinjective component $\mathscr{I}(\Lambda^\bullet)$ in $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$*-spr *ending by the hereditary sp-injective section $\mathbf{H}^-_{n_3} \hookrightarrow \mathbf{H}^-_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{H}^-_0$ of irreducible monomorphisms shown in* (4.12) (*see Fig. 1 and Remark* 6.4(b) *below*). *There is no non-zero morphism from $\mathscr{I}(\Lambda^\bullet)$ to $\mathscr{P}(\Lambda^\bullet)$.*

(b) *The Auslander–Reiten quiver $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ of $\mathrm{latt}(\Lambda^\bullet)$ is obtained from the Auslander–Reiten quiver $\Gamma((I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr})$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$*-spr *by making the identification of the section $\mathbf{P}^+_{n_3} \hookrightarrow \mathbf{P}^+_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{P}^+_0$ with the section $\mathbf{H}^-_{n_3} \hookrightarrow \mathbf{H}^-_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{H}^-_0$ and the identification of the simple projective module $P_* = e_* K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with the injective envelope $E(\mathbf{P}^+_{n_3})$ of the simple projective module $\mathbf{P}^+_{n_3} \cong e_+ K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ in $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \cong \mathrm{mod}_{\mathrm{sp}}\, K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ (see Fig.* 1).

(c) *If Λ is of finite lattice type then $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} = \mathscr{P}(\Lambda^\bullet) = \mathscr{I}(\Lambda^\bullet)$ is finite and the map $X \mapsto \mathbf{cdn}\, \mathbb{H}(X)$ establishes a bijection between the isomorphism classes of indecomposable modules in $\mathrm{latt}(\Lambda^\bullet)$ and the positive roots* (*see* (1.8)) *of the quadratic form $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z}$* (1.4) *which are distinct from the roots*

$$\varepsilon_1, \ldots, \varepsilon_{n_1+n_3}, \overline{\varepsilon}_{n_1+1}, \ldots, \overline{\varepsilon}_{n_1+n_3}, \varepsilon_*, \widehat{\varepsilon}, \widehat{\varepsilon}_{n_1+1}, \ldots, \widehat{\varepsilon}_{n_1+n_3}$$

where $\varepsilon_1, \ldots, \varepsilon_{n_1+n_3}, \overline{\varepsilon}_{n_1+1}, \ldots, \overline{\varepsilon}_{n_1+n_3}, \varepsilon_*, \varepsilon_+$ are the standard basis vectors of the free abelian group $\mathbb{Z}^{n_1+2n_3+2} = \mathbb{Z}^{n_1+n_3} \times \mathbb{Z}^{n_3+2}$, $\widehat{\varepsilon}_{n_1+s} = \varepsilon_{n_1+s} + \widehat{\varepsilon}$, $\widehat{\varepsilon} = \sum_{c \in m(\Lambda_1)} \varepsilon_c$ and $m(\Lambda_1)$ is the subset of $I_{\Lambda_1} = \{1, \ldots, n_1\}$ consisting of all elements $t \leq n_1$ such that there is no $m < t$ with ${}_m D_t = D$.

Proof. (a) Since the D-order is of the form (1.7) the poset $I' = I_{\Lambda_1}$ is linearly ordered and therefore the bound subquiver $(I^{*+}_{\Lambda^\bullet} \setminus I', \mathfrak{Z}_{\Lambda^\bullet})$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ is a union of two incomparable chains C^* and $(C')^+$. It follows like in the proof of Proposition 4.9 in [24] that every point in $(I^{*+}_{A^\bullet} \setminus I', \mathfrak{Z}_{A^\bullet})$ is separating. Then following the arguments of Bongartz [5, Theorem 2.5] we prove that there exist a unique preprojective component $\mathscr{P}(A^\bullet)$ in $(I^{*+}_{A^\bullet}, \mathfrak{Z}_{A^\bullet})$-spr starting from the hereditary projective section $\mathbf{P}^+_{n_3} \hookrightarrow \mathbf{P}^+_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{P}^+_0$ (see also [12, Section 4]).

In order to prove the existence of the preinjective component $\mathscr{I}(A^\bullet)$ we recall from [23, (2.6)] that there exists a reflection duality functor

$$(6.2) \qquad D^\bullet_S : \mathrm{mod}_{\mathrm{sp}}\, K(I^{*+}_{A^\bullet}, \mathfrak{Z}_{A^\bullet}) \longrightarrow \mathrm{mod}_{\mathrm{sp}}\, S^\bullet,$$

where $S^\bullet$ is a right two-peak reflection dual K-algebra to $S = K(I^{*+}_{A^\bullet}, \mathfrak{Z}_{A^\bullet})$ in the sense of [23, (2.6)]. Let

$$\Gamma^\bullet = rt(A^\bullet)$$

be the reflection transpose D-order in the sense of (1.5). It is easy to see that the reflection dual bound quiver $((I^{*+}_{A^\bullet})^\bullet, \mathfrak{Z}^\bullet_{A^\bullet})$ of $(I^{*+}_{A^\bullet}, \mathfrak{Z}_{A^\bullet})$ in the sense of [23, Definition 2.21] is the poset $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$ with zero-relations associated with $\Gamma^\bullet = rt(A^\bullet)$ by the formula (4.4). Note that the stratified poset $I_{\Gamma^\bullet,\sigma}$ associated with $\Gamma^\bullet$ by the formula (4.3) is just the dual form of $I_{A^\bullet,\sigma}$ (4.3).

According to [23, Corollary 2.22], there exists a K-algebra isomorphism $S^\bullet \cong K(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$ and in view of the equivalence (2.7) from the reflection duality (6.2) we derive the reflection duality functor

$$(6.3) \qquad D^\bullet : (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})\text{-spr} \longrightarrow (I^{*+}_{A^\bullet}, \mathfrak{Z}_{A^\bullet})\text{-spr},$$

By the statement (a) applied to $\Gamma^\bullet$, there exists a unique preprojective component $\mathscr{P}(\Gamma^\bullet)$ in $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$-spr ending by the hereditary projective section $\overline{\mathbf{P}}^+_{n_3} \hookrightarrow \overline{\mathbf{P}}^+_{n_3-1} \hookrightarrow \cdots \hookrightarrow \overline{\mathbf{P}}^+_0$ defined in (4.12). Since $D^\bullet$ carries the section $\overline{\mathbf{P}}^+_{n_3} \hookrightarrow \overline{\mathbf{P}}^+_{n_3-1} \hookrightarrow \cdots \hookrightarrow \overline{\mathbf{P}}^+_0$ to the hereditary sp-injective section $\mathbf{H}^-_{n_3} \hookrightarrow \mathbf{H}^-_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{H}^-_0$ in $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$-spr then $\mathscr{I}(A^\bullet) = D^\bullet(\mathscr{P}(\Gamma^\bullet))$ is the unique preinjective component with the required property. This finishes the proof of (a).

(b) Since the functor $\mathbb{H}$ is such that the diagram (4.10) is commutative then the statement (b) follows from [24, Corollary 4.29(b), Theorem 5.8] applied to the functors f^+ and f^- in (4.10), together with [18, Theorem IV; 15] applied to the composed functor

$$\mathrm{latt}(A^\bullet) \xrightarrow{\mathbb{F}} \mathrm{mod}_{\mathrm{sp}}\, R \underset{\simeq}{\xrightarrow{G}} \mathrm{mod}_{\mathrm{sp}}\, KJ_\rho$$

(4.7) and (4.9). The details are left to the reader (consult Lemma 3.8, Theorem 3.10, Corollary 3.11 and Example 4.1 in [23]).

(c) Let $R^+ = K(I^{*+}_{A^\bullet}, \mathfrak{Z}_{A^\bullet})$. Since $A^\bullet$ is of finite lattice type then according to Theorem 4.14 the category $\mathrm{mod}_{\mathrm{sp}} R^+ \cong (I^{*+}_{A^\bullet}, \mathfrak{Z}_{A^\bullet})$-spr is of finite representation type. It follows from Proposition 3.10(b) that the category $\mathrm{prin}\, R^+$ is of finite representation type.

Since, by (a), there exists a preprojective component in $\mathrm{mod}_{\mathrm{sp}} R^+$, then according to [16] there exists a preprojective component in $\mathrm{prin}\, R^+$. Moreover, it is easy to check that the Tits form $q_{\Lambda^\bullet} = q_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z}$ (1.4) is the Tits form χ_{R^+} of $\mathrm{prin}\, R^+$ in the sense of [16, Section 4], and therefore [16, Proposition 4.13] applies. It follows that the map $Z \mapsto \mathbf{cdn}\, Z$ establishes a one to one correspondence between the isomorphism classes of indecomposable modules Z in $\mathrm{prin}\, R^+$ and the positive roots of the Tits form $q_{\Lambda^\bullet}$.

Since the adjustment functor $\Theta : \mathrm{prin}\, R^+ \to \mathrm{mod}_{\mathrm{sp}} R^+ \cong (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}$ (3.2) is dense, vanishes only on the indecomposable modules ${}^\circ P_j = e_j R^+/\mathrm{soc}(e_j R^+)$, $j \in I^{*+}_{\Lambda^\bullet} \setminus \{*,+\}$ with $\mathbf{cdn}\, {}^\circ P_j = \varepsilon_j$, and $\mathbf{cdn}\, Z = \mathbf{cdn}\, \Theta(Z)$ for every indecomposable module Z in $\mathrm{prin}\, R^+$ which is not isomorphic to a module ${}^\circ P_j$ then the map $Y \mapsto \mathbf{cdn}\, Y$ establishes a one to one correspondence between the isomorphism classes of indecomposable modules in $\mathrm{mod}_{\mathrm{sp}} R^+ \cong (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}$ and the positive roots of the Tits form $q_{\Lambda^\bullet}$ which are distinct from the roots $\varepsilon_1, \dots, \varepsilon_{n_1+n_3}, \overline{\varepsilon}_{n_1+1}, \dots, \overline{\varepsilon}_{n_1+n_3}$ being coordinate vectors of the modules ${}^\circ P_j$ for $j \in I^{*+}_{\Lambda^\bullet} \setminus \{*,+\}$.

By Theorem 4.14, the only indecomposable objects of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr, up to isomorphism, being not in the image of the functor $\mathbb{H}$ are the modules $\mathbf{H}^-_{n_3} \hookrightarrow \mathbf{H}^-_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{H}^-_0$ and $e_* R^+$. Since their coordinate vectors are just the roots $\varepsilon_*, \widehat{\varepsilon}, \widehat{\varepsilon}_{n_1+1}, \dots, \widehat{\varepsilon}_{n_1+n_3}$ respectively, then (c) follows from Theorem 4.14(b) and the discussion above. This finishes the proof of the theorem. □

Remark 6.4. Assume that $\Lambda^\bullet$ is a subamalgam D-order as in Theorem 6.1. We describe now a computer accessible algorithm constructing $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ of $\mathrm{latt}(\Lambda^\bullet)$ in case $\Lambda^\bullet$ is of finite lattice type. The procedure is illustrated below in Example 6.6.

(a) It follows from Theorem 6.1 and [24, Theorem 4.30] that if $\Lambda^\bullet$ is of finite lattice type then the Auslander-Reiten quiver $\Gamma((I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}) = \mathscr{P}(\Lambda^\bullet) = \mathscr{I}(\Lambda^\bullet)$ is finite, the part $\mathscr{R}$ is empty (see Fig. 1), every indecomposable object X in $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr is uniquely determined by its coordinate vector $\mathbf{cdn} X$ (see [26, Corollary 3.2]), $\mathrm{End}(X) \cong K$ and $\mathrm{Ext}_{K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}(X,X) = 0$.

(b) The preprojective component $\mathscr{P}(\Lambda^\bullet)$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr can be constructed by a slight modification of the algorithm given in Sections 11.9–11.11 of [25] for J^*-spr, where J^* is a one-peak poset, and the algorithm described in [12, Section 4] for I-spr, where I is an $\widetilde{\mathbb{A}}$-free multi-peak poset (see also [10]).

It was shown in the proof of Theorem 6.1 that the preinjective component $\mathscr{I}(\Lambda^\bullet)$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr can be obtained from the preprojective component $\mathscr{P}(\Gamma^\bullet)$ of $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$-spr by applying the reflection duality functor (6.3).

(c) If $\Lambda^\bullet$ is of finite lattice type then the Auslander–Reiten quiver $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ of $\mathrm{latt}(\Lambda^\bullet)$ can be constructed from the Auslander–Reiten quiver $\Gamma((I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}) = \mathscr{P}(\Lambda^\bullet)$ (see above) by a simple glueing described in Theorem 6.1 (b).

Remark 6.5. It would be interesting and useful for applications to present an explicit form of the reduction functor $\mathbb{H} : \mathrm{latt}(\Lambda) \to (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr (see 4.11).

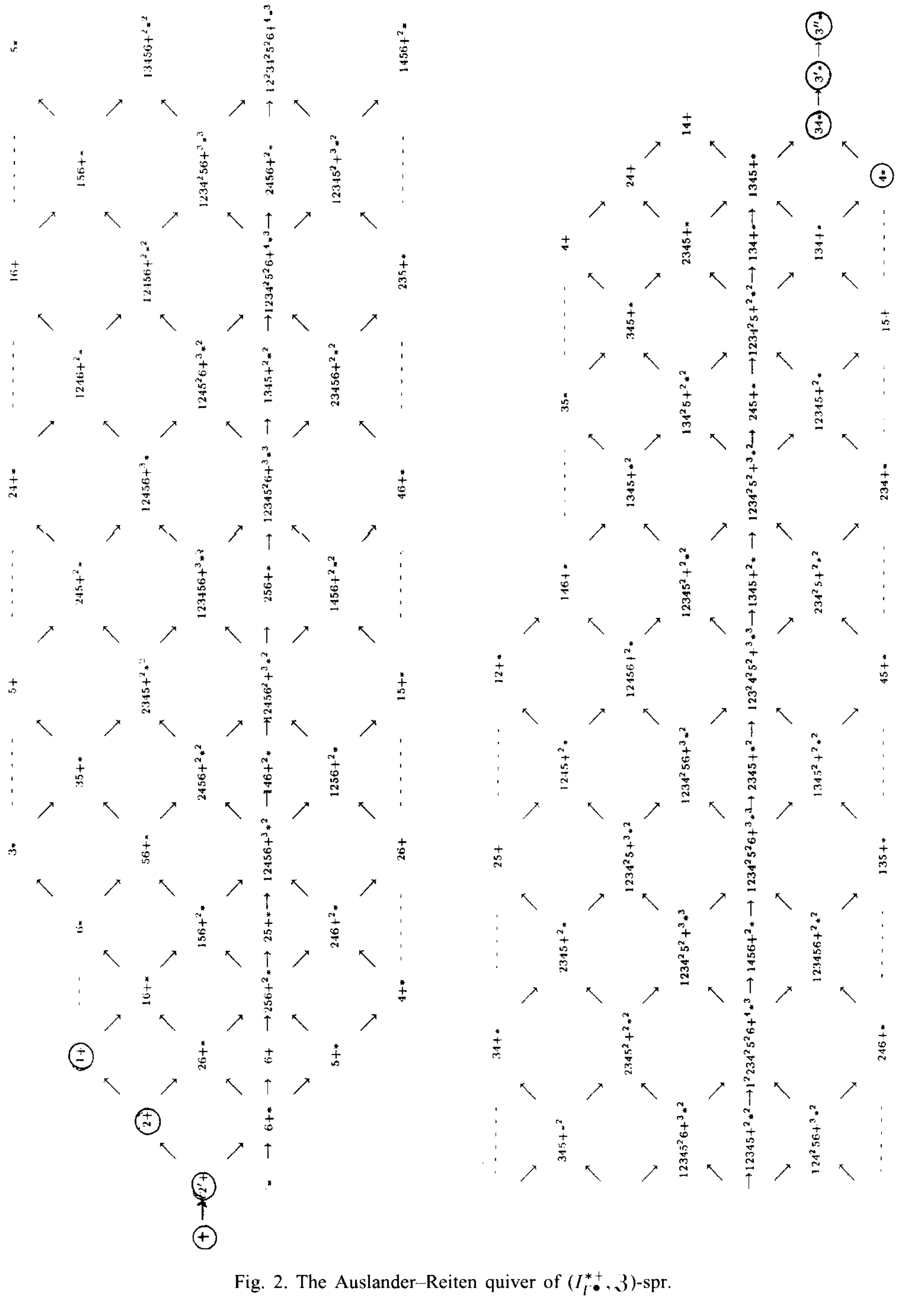


Fig. 2. The Auslander–Reiten quiver of $(I_{I\bullet}^{*+}, \mathfrak{Z})$-spr.

Example 6.6. Let $\Gamma_\bullet$ be the subamalgam from Example 4.5. It follows from Theorem 1.6 that $\Gamma_\bullet$ is of finite lattice type. We shall show, by applying the method described in Remark 6.4, that the number of the isomorphism classes of indecomposable lattices in $\mathrm{latt}(\Gamma_\bullet)$ is equal to 96 and we describe the Auslander–Reiten quiver $\Gamma(\mathrm{latt}(\Gamma^\bullet))$ of $\mathrm{latt}(\Gamma_\bullet)$.

For a computational reason we renumber the poset $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$ with zero-relations of Example 4.5 as follows

$$(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet}): \quad \begin{tikzcd} 3'' \arrow[r] & 3' \arrow[r] \arrow[d] & 3 \arrow[dr] \arrow[ddrr, dashed, no head] & & & \\ & 4 \arrow[r] & 5 \arrow[r] \arrow[rr, dashed, no head, bend right] & 6 \arrow[r] \arrow[dr] & * & \\ & & 1 \arrow[r] & 2 \arrow[r] & 2' \arrow[r] & + \end{tikzcd}$$

where the dotted line means a zero-relation. It follows from Remarks 6.4 that the Auslander–Reiten quiver of $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$-spr is that one presented in Fig. 2, where we write the coordinate vectors $v = \mathbf{cdn}\, V$ instead of indecomposable representations V of $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$, and we use the exponential notation of coordinate vectors introduced in [25, 11.88], that is, the vector $v = \mathbf{cdn}\, V = (v_1, \dots, v_t) \in \mathbb{N}^{I^{*+}_{\Gamma^\bullet}}$ is written in the form

$$v = \mathbf{cdn}\, V = 1^{v_1} 2^{v_2} \dots t^{v_t}$$

where we omit j^{v_j} if $v_j = 0$, and we set $j^{v(j)} = j$ if $v(j) = 1$. By [16, Proposition 4.13] the category $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$-spr has exactly 101 indecomposable objects up to isomorphism and the number of positive roots of the Tits quadratic form $q_{(\Gamma^\bullet, \mathfrak{Z}_{\Gamma^\bullet})}$ equals 110.

The Auslander–Reiten quiver $\Gamma(\mathrm{latt}(\Gamma^\bullet))$ of $\mathrm{latt}(\Gamma_\bullet)$ is obtained from that one in Fig. 2 by making the identification of the hereditary projective encircled section $+ - 2'+ \to 2+ \to 1+$ from the beginning of $\Gamma((I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})\text{-spr})$ with the hereditary sp-injective encircled section $4* \to 34* \to 3'* \to 3''*$ from the end of $\Gamma((I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})\text{-spr})$, and the identification of the vertex $* = \mathbf{cdn}\, P_*$ with the vertex $14+ = \mathbf{cdn}\, E(P_+)$. Consequently the D-order $\Gamma^\bullet$ is of finite lattice type and the number of the isomorphism classes of indecomposable $\Gamma^\bullet$-lattices equals 96.

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